EXAM 2: SOLUTION

1. Suppose a dealer’s profit (in units of $5000) is considered a r.v. $X$ having probability density function

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$ 

(a) Compute $E(X)$. (2 points)

(b) Compute $\text{var}(X)$. (2 points)

(c) Compute $E(3X^2 - 7)$. (2 points)

Answer.

(a) $E(X) = \int_0^1 x2(1-x)dx = 2\int_0^1 (x-x^2)dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$.

(b) $E(X^2) = \int_0^1 x^22(1-x)dx = 2\int_0^1 (x^2-x^3)dx = 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{6}$.

Hence $\text{var}(X) = E(X^2) - (E(X))^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$.

(c) $E(3X^2 - 7) = 3E(X^2) - 7 = 3\left(\frac{1}{6}\right) - 7 = -\frac{13}{2}$.

2. Suppose $X$ and $Y$ are discrete random variables with joint probability mass function:

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<tr>
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<th>$X$</th>
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<tr>
<td></td>
<td>2</td>
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<tr>
<td>1</td>
<td>0.10</td>
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<tr>
<td>3</td>
<td>0.20</td>
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<td>5</td>
<td>0.10</td>
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(a) Find the marginal distribution of $X$. (2 points)

(b) Find the marginal distribution of $Y$. (2 points)

(c) Find the conditional distribution of $Y$ given that $X = 4$. (2 points)

(d) Find $E(Y \mid X = 4)$. (2 points)

(e) Find $\text{cov}(X, Y)$. How strong is the linear dependence between $X$ and $Y$? (4 points)

Answer.

(a) $f_X(2) = 0.1 + 0.2 + 0.1 = 0.4$, $f_X(4) = 0.15 + 0.3 + 0.15 = 0.6$.

(b) $f_Y(1) = 0.1 + 0.15 = 0.25$, $f_Y(3) = 0.2 + 0.3 = 0.5$, $f_Y(5) = 0.1 + 0.15 = 0.25$.

(c) $f_{Y \mid X}(1\mid 4) = P(Y = 1 \mid X = 4) = \frac{f_Y(1) f_X(4)}{f_X(4)} = \frac{0.15}{0.6} = 0.25$, $f_{Y \mid X}(3\mid 4) = \frac{f_Y(3) f_X(4)}{f_X(4)} = \frac{0.3}{0.6} = 0.5$,

$\quad f_{Y \mid X}(5\mid 4) = \frac{f_Y(5) f_X(4)}{f_X(4)} = \frac{0.15}{0.6} = 0.25$.

(d) $E(Y \mid X = 4) = \sum_y y f_{Y \mid X}(y \mid 4) = 1(0.25) + 3(0.5) + 5(0.25) = 3$.
(e) \( E(X) = 2(0.4) + 4(0.6) = 3.2 \), \( E(Y) = 1(0.25) + 3(0.5) + 5(0.25) = 3 \),
\( E(XY) = 2(0.1) + 6(0.2) + 10(0.1) + 4(0.15) + 12(0.3) + 20(0.15) = 9.6 \).
Hence \( \text{cov}(X, Y) = E(XY) - E(X)E(Y) = 9.6 - 3.2(3) = 0 \).
There is no linear dependence between \( X \) and \( Y \).

3. A particular type of tennis racket comes in a mid-size version and an oversize version. Sixty percent of all customers at a certain store want the oversize version (the rest wants the mid-size version). Suppose ten customers who want this type of racket are randomly selected. Compute
(a) The probability that at least six want the oversize version? (2 points)
(b) The mean and variance of the number of customers that want the oversize version. (2 points)
(c) The probability that all customers can get the version they want from the current stock which currently has seven rackets of each version. (3 points)

Answer.

If \( X \) = number of customers that want the oversize version racket out of ten, then \( X \sim \text{bin}(10, 0.6) \).
(a) \( P(X \geq 6) = 1 - P(X \leq 5) = 1 - 0.367 = 0.633 \)
(b) \( E(X) = np = 10(0.6) = 6 \) and \( V(X) = np(1 - p) = 6(0.4) = 2.4 \).
(c) \( P(3 \leq X \leq 7) = P(X \leq 7) - P(X \leq 2) = 0.833 - 0.012 = 0.821 \)

4. What is the probability that a waitress will refuse to serve alcoholic beverages to 2 minors if she randomly checks the IDs of 5 students from among 9 students of which 4 are minors. (2 points)

Answer.

\[
p = \frac{\binom{4}{2} \binom{5}{3}}{\binom{9}{5}} = 0.4762
\]