Chapter 11
Diversification and Asset Allocation

Concept Questions

1. Based on market history, the average annual standard deviation of return for a single, randomly chosen stock is about 50 percent. The average annual standard deviation for an equally-weighted portfolio of many stocks is about 20 percent.

2. If the returns on two stocks are highly correlated, they have a strong tendency to move up and down together. If they have no correlation, there is no particular connection between the two. If they are negatively correlated, they tend to move in opposite directions.

3. An efficient portfolio is one that has the highest return for its level of risk.

4. True. Remember, portfolio return is a weighted average of individual returns.

5. False. Remember the principle of diversification.

6. Because of the effects of diversification, an investor will never receive the highest return possible from a single asset. However, the investor will also never receive the lowest return. More importantly, even though an investor does give up the potential “home run” investment, the reduction in return is more than offset by the reduction in risk. In other words, you give up a little return for a lot less risk.

7. You know your current portfolio is the minimum variance portfolio (or below). Below and to the right of the minimum variance portfolio, as you add more of the lower risk asset, the standard deviation of your portfolio increases and the expected return decreases.

8. The importance of the minimum variance portfolio is that it determines the lower bond of the efficient frontier. While there are portfolios on the investment opportunity set to the right and below the minimum variance portfolio, they are inefficient. That is, there is a portfolio with the same level of risk and a higher return. No rational investor would ever invest in a portfolio below the minimum variance portfolio.

9. False. Individual assets can lie on the efficient frontier depending on its expected return, standard deviation, and correlation with all other assets.

10. If two assets have zero correlation and the same standard deviation, then evaluating the general expression for the minimum variance portfolio shows that x = ½; in other words, an equally-weighted portfolio is minimum variance.

Core Questions

1. \(0.2(-0.10) + 0.6(0.20) + 0.2(0.30) = 16\%\)

2. \(0.2(-0.10 - 0.16)^2 + 0.4(0.20 - 0.16)^2 + 0.2(0.30 - 0.16)^2 = 0.00184; s = 13.56\%\)
3. \[(1/3)(-0.10) + (1/3)(0.20) + (1/3)(0.30) = 13.333\% \]
\[(1/3)(-0.10 - 0.13)^2 + (1/3)(0.20 - 0.13)^2 + (1/3)(0.30 - 0.13)^2 = 0.02889; s = 16.997\% \]

4. 

Calculating Expected Returns

<table>
<thead>
<tr>
<th>Roll</th>
<th>Probability of State of Economy</th>
<th>Return if State Occurs</th>
<th>Product (2) × (3)</th>
<th>Ross</th>
<th>Probability of State of Economy</th>
<th>Return if State Occurs</th>
<th>Product (2) × (5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bust</td>
<td>0.40</td>
<td>-10%</td>
<td>-0.04</td>
<td>40%</td>
<td>0.40</td>
<td>-10%</td>
<td>0.16</td>
</tr>
<tr>
<td>Boom</td>
<td>0.60</td>
<td>50%</td>
<td>0.30</td>
<td>10%</td>
<td>0.60</td>
<td>50%</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td><strong>E(R)</strong> = 26%</td>
<td></td>
<td></td>
<td></td>
<td><strong>E(R)</strong> = 22%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. 

<table>
<thead>
<tr>
<th>Roll</th>
<th>Probability of State of Economy</th>
<th>Return Deviation from Expected Return</th>
<th>Squared Return Deviation Product (2) × (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bust</td>
<td>.40</td>
<td>-.36</td>
<td>.1296</td>
</tr>
<tr>
<td>Boom</td>
<td>.60</td>
<td>.24</td>
<td>.0576</td>
</tr>
<tr>
<td></td>
<td><strong>s^2</strong> = .0864</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ross</th>
<th>Probability of State of Economy</th>
<th>Return Deviation from Expected Return</th>
<th>Squared Return Deviation Product (2) × (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bust</td>
<td>.40</td>
<td>.18</td>
<td>.0324</td>
</tr>
<tr>
<td>Boom</td>
<td>.60</td>
<td>-.12</td>
<td>.0144</td>
</tr>
<tr>
<td></td>
<td><strong>s^2</strong> = .0216</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Taking square roots, the standard deviations are 29.39% and 14.70%.

6. 

Expected Portfolio Return

<table>
<thead>
<tr>
<th>State of Economy</th>
<th>Probability of State of Economy</th>
<th>Portfolio Return if State Occurs</th>
<th>Product (2) × (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bust</td>
<td>.40</td>
<td>20%</td>
<td>.080</td>
</tr>
<tr>
<td>Boom</td>
<td>.60</td>
<td>26%</td>
<td>.156</td>
</tr>
<tr>
<td></td>
<td><strong>E(R_p)</strong> = 23.6%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
7. Calculating Portfolio Variance

<table>
<thead>
<tr>
<th>(1) State of Economy</th>
<th>(2) Probability of State of Economy</th>
<th>(3) Portfolio Return if State Occurs</th>
<th>(4) Squared Deviation from Expected Return</th>
<th>(5) Product (2) \times (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bust</td>
<td>.40</td>
<td>.05</td>
<td>.0392</td>
<td>.0157</td>
</tr>
<tr>
<td>Boom</td>
<td>.60</td>
<td>.38</td>
<td>.0174</td>
<td>.0105</td>
</tr>
</tbody>
</table>

\[ s^2_p = 0.0262 \]
\[ s_p = 16.17\% \]

8. \( E[R_A] = .20(.06) + .60(.07) + .20(.11) = 7.60\% \)
\( E[R_B] = .20(-.2) + .60(.13) + .20(.33) = 10.40\% \)
\( \sigma_A^2 = .20(.06-.0760)^2 + .60(.07-.0760)^2 + .20(.11-.0760)^2 = .000304; \quad \sigma_A = [.000304]^{1/2} = .01744 \)
\( \sigma_B^2 = .20(-.2-.1040)^2 + .60(.13-.1040)^2 + .20(.33-.1040)^2 = .029104; \quad \sigma_B = [.029104]^{1/2} = .17060 \)

9. a. boom: \( E[R_p] = .30(.3) + .40(.45) + .30(.33) = .3690 \)
good: \( E[R_p] = .30(.12) + .40(.10) + .30(.15) = .1210 \)
poor: \( E[R_p] = .30(.01) + .40(-.15) + .30(-.05) = -.0720 \)
bust: \( E[R_p] = .30(-.06) + .40(-.30) + .30(-.09) = -.1650 \)
\( E[R_p] = .20(.3690) + .40(.1210) + .30(-.0720) + .10(-.1650) = .0841 \)
b. \( \sigma_p^2 = .20(.3690 -.0841)^2 + .40(.1210 -.0841)^2 + .30(-.0720 -.0841)^2 + .10(-.1650 -.0841)^2 \)
\( \sigma_p^2 = .03029; \quad \sigma_p = [.03029]^{1/2} = .1741 \)

10. Notice that we have historical information here, so we calculate the sample average and sample standard deviation (using \( n - 1 \)) just like we did in Chapter 1. Notice also that the portfolio has less risk than either asset.

<table>
<thead>
<tr>
<th>Year</th>
<th>Stock A</th>
<th>Stock B</th>
<th>Portfolio AB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1999</td>
<td>18%</td>
<td>65%</td>
<td>30.80%</td>
</tr>
<tr>
<td>2000</td>
<td>40</td>
<td>-30</td>
<td>12.00</td>
</tr>
<tr>
<td>2001</td>
<td>-15</td>
<td>45</td>
<td>9.00</td>
</tr>
<tr>
<td>2002</td>
<td>20</td>
<td>2</td>
<td>12.80</td>
</tr>
<tr>
<td>2003</td>
<td>4</td>
<td>20</td>
<td>10.40</td>
</tr>
</tbody>
</table>

Avg returns: 13.40% 17.40% 15.00%
Std deviations: 20.42% 32.85% 8.95%
Intermediate Questions

11. Boom: \( .30(15\%) + .40(18\%) + .30(20\%) = 17.70\% \)
    Bust: \( .30(10\%) + .40(0\%) + .30(–10\%) = 0.00\% \)

\[ E(R_p) = .7(1.1770) + .3(0.00) = 12.39\% \]

\[ \sigma^2_p = .7(1.1770 – 1.1239)^2 + .3(0.00 – 1.1239)^2 = .0066; \ s_p = 8.11\% \]

12. \( E(R_p) = .4(.19) + .6(.11) = 14.20\% \)

\[ \sigma^2_p = .4^2(.65^2) + .6^2(.45^2) + 2(.4)(.6)(.65)(.45)(.20) = .16858; \ s_p = 41.06\% . \]

13. \( \sigma^2_p = .4^2(.65^2) + .6^2(.45^2) + 2(.4)(.6)(.65)(.45)(1.0) = .28090; \ s_p = 53.00\% \)

\[ \sigma^2_p = .4^2(.65^2) + .6^2(.45^2) + 2(.4)(.6)(.65)(.45)(0.0) = .14050; \ s_p = 37.48\% \]

\[ \sigma^2_p = .4^2(.65^2) + .6^2(.45^2) + 2(.4)(.6)(.65)(.45)(–1.0) = .00010; \ s_p = 1.00\% \]

14. \( w_{3\text{Doors}} = \frac{.45^2 - .65 \times .45 \times .20}{.65^2 + .45^2 - 2 \times .65 \times .45 \times 20} = 0.2835; \ w_S = (1 – 0.2835) = .7165 \)

\[ E(R_p) = .2835(.19) + .7165(.11) = 13.27\% \]

\[ \sigma^2_p = .2835^2(.65^2) + .7165^2(.45^2) + 2(.2835)(.7165)(.65)(.45)(.20) = .16168 \]

\[ s_p = 40.21\% \]

15. Risk and Return with Stocks and Bonds

<table>
<thead>
<tr>
<th>Portfolio Weights</th>
<th>Expected Return</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stocks</td>
<td>Bonds</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>0.00</td>
<td>13.00%</td>
</tr>
<tr>
<td>0.80</td>
<td>0.20</td>
<td>11.60%</td>
</tr>
<tr>
<td>0.60</td>
<td>0.40</td>
<td>10.20%</td>
</tr>
<tr>
<td>0.40</td>
<td>0.60</td>
<td>8.80%</td>
</tr>
<tr>
<td>0.20</td>
<td>0.80</td>
<td>7.40%</td>
</tr>
<tr>
<td>0.00</td>
<td>1.00</td>
<td>6.00%</td>
</tr>
</tbody>
</table>

16. \( w_D = \frac{.78^2 - .54 \times .78 \times .40}{.54^2 + .78^2 - 2 \times .54 \times .78 \times .40} = 0.6281; \ w_1 = (1 – 0.6281) = .3719 \)

17. \( E(R_p) = .6281(.17) + .3719(.15) = 16.26\% \)

\[ \sigma^2_p = .6281^2(.54^2) + .3719^2(.78^2) + 2(.6281)(.3719)(.54)(.78)(–.40) = .12048 \]

\[ s_p = 34.71\% \]
18. \[ w_K = \frac{.09^2 - .34 \times .09 \times .08}{.34^2 + .09^2 - 2 \times .34 \times .09 \times .08} = 0.0476; \quad w_L = (1 - 0.0476) = .9524 \]
\[ E(R_p) = 0.0476(.17) + .9524(.06) = 6.52\% \]
\[ \sigma_p^2 = .0476^2(.34^2) + .9524^2(.09^2) + 2(.0476)(.9524)(.34)(.09)(.08) = .00783 \]
\[ s_p = 8.85\% \]

19. \[ w_{\text{Bruin}} = \frac{.54^2 - .48 \times .54 \times .25}{.48^2 + .54^2 - 2 \times .48 \times .54 \times .25} = 0.5780; \quad w_{\text{Wildcat}} = (1 - 0.5780) = .4220 \]
\[ E(R_p) = .5780(.18) + .4220(.15) = 16.73\% \]
\[ \sigma_p^2 = .5780^2(.48^2) + .4220^2(.54^2) + 2(.5780)(.4220)(.48)(.54)(.25) = .16051 \]
\[ s_p = 40.06\% \]

20. \[ E(R) = .30(14\%) + .50(16\%) + .20(13\%) = 14.80\% \]
\[ \sigma_p^2 = .30^2(.53^2) + .50^2(.67^2) + .20^2(.48^2) + 2(.30)(.50)(.53)(.67)(.30) + 2(.30)(.20)(.53)(.48)(.20) + 2(.50)(.20)(.67)(.48)(.05) = 0.18800 \]
\[ s_p = 43.36\% \]

21. \[ w_j = \frac{.15^2 - .58 \times .15 \times .30}{.58^2 + .15^2 - 2 \times .58 \times .15 \times .30} = \frac{-0.0036}{0.3067} = -0.0117; \quad w_S = (1 - (-0.0117)) = 1.0117 \]
\[ \sigma_p^2 = (-0.0117)^2(.58^2) + (1.0117)^2(.15^2) + 2(.58)(.15)(1.0117)(-0.0117)(.30) = .02246 \]
\[ s_p = (.02246)^{1/2} = 14.99\% \]
\[ E(R_p) = -0.0117(.16) + 1.0117(.08) = 7.91\% \]

Even though it is possible to mathematically calculate the standard deviation and expected return of a portfolio with a negative weight, an explicit assumption is that no asset can have a negative weight. The reason this portfolio has a negative weight in one asset is the relatively high correlation between the two assets. If you look at the investment opportunity sets in the chapter, you will notice that as the correlation decreases, the investment opportunity set bends further backwards. However, for a portfolio with a correlation of +1, there is no minimum variance portfolio with a variance lower than the lowest variance asset. This implies there is some necessary level of correlation to make the minimum variance portfolio have a variance lower than the lowest variance asset. The formula to determine if there is a minimum variance portfolio with a variance less than the lowest variance asset is: \[ \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} > \rho \]. In this case, \[ \frac{.15}{.58} = .258 < .30 \] so there is no minimum variance portfolio with a variance lower than the lowest variance asset assuming non-negative asset weights.

22. Look at \( \sigma_p^2 \):
\[ \sigma_p^2 = (x_A \times s_A + x_B \times s_B)^2 = x_A^2 \times \sigma_A^2 + x_B^2 \times \sigma_B^2 + 2 \times x_A \times x_B \times \sigma_A \times \sigma_B \times 1, \] which is precisely the expression for the variance on a two-asset portfolio when the correlation is +1.
23. Look at $\sigma_p^2$:

$$\sigma_p^2 = (x_A \times s_A - x_B \times s_B)^2$$
$$= x_A^2 \times \sigma_A^2 + x_B^2 \times \sigma_B^2 + 2 \times x_A \times x_B \times \sigma_A \times \sigma_B \times (-1),$$
which is precisely the expression for the variance on a two-asset portfolio when the correlation is $-1$.

24. From the previous question, with a correlation of $-1$:

$$s_p = x_A \times s_A - x_B \times s_B$$
$$= x \times s_A - (1 - x) \times s_B$$
Set this to equal zero and solve for $x$ to get:

$$0 = x \times s_A - (1 - x) \times s_B$$
$$x = \frac{s_B}{s_A + s_B}$$
This is the weight on the first asset.

25. Let $\rho$ stand for the correlation, then:

$$\sigma_p^2 = x_A^2 \times \sigma_A^2 + x_B^2 \times \sigma_B^2 + 2 \times x_A \times x_B \times \sigma_A \times \sigma_B \times \rho$$
$$= x^2 \times s_A^2 + (1 - x)^2 \times s_B^2 + 2 \times x \times (1 - x) \times s_A \times s_B \times \rho$$
Take the derivative with respect to $x$ and set equal to zero:

$$\frac{ds_p^2}{dx} = 2 \times x \times s_A^2 - 2 \times (1 - x) \times s_B^2 + 2 \times s_A \times s_B \times \rho - 4 \times x \times s_A \times s_B \times \rho = 0$$
Solve for $x$ to get the expression in the text.