Chapter 11 - Diversification and Asset Allocation

Looking Forward vs. Looking Backwards

Looking Back: \( \overline{R_{A}} = \text{return on } A \text{ in period } i \)

\[
\overline{R_A} = \frac{R_{1A} + R_{2A} + R_{3A} + \cdots + R_{nA}}{N}
\]

Can also write as:

\[
\overline{R_A} = \frac{1}{N} R_{1A} + \frac{1}{N} R_{2A} + \frac{1}{N} R_{3A} + \cdots + \frac{1}{N} R_{nA}
\]

Looking Forward

Let \( \Pi_i \) = the probability that \( i \) will occur

\( i \) = "State of Nature"; often state of economy

\( R_{iA} \) = return in state \( i \) for stock \( A \)

\[
E(R_A) = \Pi_1 R_{1A} + \Pi_2 R_{2A} + \cdots + \Pi_n R_{nA}
\]

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\[
E(R) = 0.1(15\%) + 0.6(13\%) + 0.3(7\%)
\]

\[
= 1.5\% + 7.8\% + 2.1\%
\]

\[
= 11.4\%
\]

Risk: Looking Back

\[
\text{Var} (\sigma^2) = \frac{1}{N-1} (R_{1A} - \overline{R_A})^2 + \frac{1}{N-1} (R_{2A} - \overline{R_A})^2 + \cdots + \frac{1}{N-1} (R_{nA} - \overline{R_A})^2
\]
Looking Forward

\[ \text{Var}(\sigma^2) = \pi_1 \left( R_{1\text{A}} - E(R_{\text{A}}) \right)^2 + \pi_2 \left( R_{2\text{A}} - E(R_{\text{A}}) \right)^2 + \cdots + \pi_n \left( R_{n\text{A}} - E(R_{\text{A}}) \right)^2 \]

\[ \sigma = \text{standard deviation} = \sqrt{\text{Var}} = \sqrt{\sigma^2} \]

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<table>
<thead>
<tr>
<th>STATE</th>
<th>PROB</th>
<th>RETURN</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{3} )</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{3} )</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{3} )</td>
<td>50</td>
</tr>
</tbody>
</table>

\[ E(R) = \frac{1}{3}(0) + \frac{1}{3}(25) + \frac{1}{3}(50) = 25 \]

\[ \sigma^2 = \frac{1}{3}(0 - 25)^2 + \frac{1}{3}(25 - 25)^2 + \frac{1}{3}(50 - 25)^2 \]

\[ = \frac{1}{3}(625) + \frac{1}{3}(0) + \frac{1}{3}(625) \]

\[ = 416.67 \]

\[ \sigma = \sqrt{416.67} \approx 20.41 \]

Note: If the \( \pi_i \) are equal to each other, then we can use stat features on our calculator to compute expected return and standard deviation.
Portfolio - A group of assets such as stocks and bonds

Portfolio Weight - Proportion of a portfolio invested in a specific asset
\[ \sum_{i=1}^{N} x_i = 1 \]

Investors Decision - The one decision an investor has is choosing his or her portfolio weights. A portfolio weight of 0 means I have no position in that asset (which is true for most assets). A negative portfolio weight represents a short position in that asset, which could be, for example, a margin loan.

Expected return on a portfolio: \( E(R_p) \)
\[ E(R_p) = \sum_{i=1}^{N} x_i E(R_i) \]

Where: \( N \) = # of assets in portfolio
\[ x_i = P_{i} \times W_{i} \]
and \[ \sum_{i=1}^{N} x_i = 1 \]
For a 2-asset portfolio we can write
\[ E(R_p) = \pi_1 E(R_1) + \pi_2 E(R_2) \]
or because \( \pi_1 + \pi_2 = 1 \) \( \Rightarrow \pi_2 = 1 - \pi_1 \)
\[ E(R_p) = \pi_1 E(R_1) + (1 - \pi_1) E(R_2) \]

Risk of a Portfolio

- One might think that a similar looking formula could be used to compute the standard deviation of a portfolio.
- However, the risk of the assets in the portfolio may "cancel" each other out i.e. the "downs" of one asset may be balanced by the "ups" of another.
Principle of Diversification

Spreading investment risk across a number of assets will eliminate some but not all of your risk.

\[ \text{Total Risk} = \text{Diversifiable Risk} + \text{Non Diversifiable Risk} \]

\[ \text{Correlation} = \text{The tendency of the returns of two assets to move together} \]
\[ -1 \leq \text{Corr}(R_A, R_B) \leq 1 \]

A negative correlation means things tend to move in opposite directions.

A zero correlation means there is no tendency to move together.
Portfolio Std. Deviation

The variance for a 2-asset portfolio can be written

\[ \sigma_p^2 = \sigma_1^2 \sigma_2^2 + \sigma_2^2 \sigma_1^2 + 2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \text{Corr}(R_1, R_2) \]

The std. deviation is the square root of the variance.

\[ \sigma_p = \sqrt{\sigma_p^2} \]

An example of applying the expected return and std. deviation formulas using the data for Table 11.9.

Compute the expected return and standard deviation for a 70% stock, 30% bond portfolio, with the following data:

\[ E(R_s) = 0.12 \quad \sigma_s = 0.15 \]
\[ E(R_b) = 0.06 \quad \sigma_b = 0.10 \]
\[ \text{Corr}(R_s, R_b) = 0.10 \]

\[ E(R_p) = \sigma_1 E(R_1) + \sigma_2 E(R_2) \]
\[ = 0.7(0.12) + 0.3(0.06) \]
\[ = 0.084 + 0.018 \]
\[ = 0.102 \approx 10.2\% \]
\[ \sigma_p^2 = (0.7)^2 (0.15)^2 + (0.3)^2 (0.10)^2 + 2(0.7)(0.3)(0.15)(0.10)(0.1) \]
\[ = 0.011025 + 0.000900 + 0.000630 \]
\[ = 0.012555 \]
\[ \sigma_p = \sqrt{0.012555} = 0.112049 = 11.20\% \]

Minimum Variance Portfolio

- All pth's have a variance
- One particular choice of asset allocation is to choose the portfolio weights that will result in the minimum risk portfolio.

For a 2 asset portfolio

\[ \sigma_p = \left[ x^2 \sigma_A^2 + (1-x)^2 \sigma_B^2 + 2x(1-x) \sigma_A \sigma_B \text{corr}(\mathbf{R}_A, \mathbf{R}_B) \right]^{1/2} \]

\[ \frac{d \sigma_p}{dx} = \frac{1}{2} \left[ \text{ditto} \right]^{-1/2} \text{(derivative of ditto)} \]

then solve for \( x^* \)

\[ x^* = \frac{\sigma_B^2 - \sigma_A \sigma_B \text{corr} (\mathbf{R}_A, \mathbf{R}_B)}{\sigma_A^2 - \sigma_B^2 - 2 \sigma_A \sigma_B \text{corr} (\mathbf{R}_A, \mathbf{R}_B)} \]

where \( x^* \) is the portfolio weight for asset A which yields the lowest risk portfolio.
If \( \text{Cov}(R_A, R_B) = 0 \) then 
\[
\sigma^* = \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}
\]

Also: if \( \text{Cov}(R_A, R_B) = -1 \) there exist a portfolio with \( \sigma_p = 0 \)
i.e. when
\[
\sigma^* = \frac{\sigma_B^2 + \sigma_A \sigma_B}{\sigma_A^2 + \sigma_B^2 + 2 \sigma_A \sigma_B}
\]

**Efficient Portfolio**

A portfolio that offers the highest return for its level of risk.

\( E(R_p) \)

min Variance Ptf

Non efficient or dominated portfolios

\( \Gamma_p \)
Generalizing to $N$ assets

\[ E(R_p) \]

Called the **Markowitz Efficient Frontier**

**Markowitz Efficient Frontier** is the set of all efficient portfolios.