Time Value of Money (TVM)
Discounted Cash Flow (DCF)

\[ V_{t+1} = V_t + V_t \left( \frac{1}{1 + i_{t+1}} \right) \]

This equation links value between periods where:

- \( V_t \): Value at time \( t \)
- \( V_{t+1} \): Value at time \( t+1 \)
- \( i_{t+1} \): Interest rate earned over the \( t+1 \) period

\[ t \quad \quad t+1 \quad t+2 \]

\[ V_t \quad \quad i_{t+1} \quad V_{t+1} \]

By analogy, we can also write:

\[ V_{t+2} = V_{t+1} + V_{t+1} \left( \frac{1}{1 + i_{t+2}} \right) \]
By algebra, I can also write

\[ V_{t+2} = V_{t+1} (1 + i_{t+2}) \]

Also note that \( V_{t+1} = V_t (1 + i_{t+1}) \)

So I can write

\[ V_{t+2} = V_t (1 + i_{t+1})(1 + i_{t+2}) \]

Let \( t = 2000 \) (i.e. your year 2000)

Then

\[ V_{2002} = V_{2000} (1 + i_{2001})(1 + i_{2002}) \]

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You take out a loan for \$100,000
that charges 1% the first month
and 0.5% the second month.
What will your balance be, assuming
you make no payments (after 2 months)
If we allow this process to continue for \( N \) periods we can write

\[
V_{t+N} = V_t \left(1+i_{t+1} \right) \left(1+i_{t+2} \right) \ldots \left(1+i_{t+N} \right)
\]

\( N \) terms

1. If the interest rate was constant \( i \) for \( t \) to \( t+N \)

\[
V_{t+N} = V_t (1+i)^N
\]

2. If \( t=0 \) (i.e., the present)

\[
V_N = V_0 (1+i)^N
\]

or also written

\[
FV = PV (1+i)^N
\]

This is the FV formula for a constant interest rate.
The formula above states that:

\[ FV \quad \text{increases on} \]

- \( PV \uparrow \)
- \( i \uparrow \)
- \( N \uparrow \)

From \( FV \quad PV \quad (1+i)^n \)
we can divide both sides by \( (1+i)^n \):

\[
\frac{FV}{(1+i)^n} \quad \frac{PV}{(1+i)^n}
\]

\[
FV \quad PV
\]

which is the standard single sum discounting equation.

This shows that \( PV \uparrow \)

If \( FV \uparrow \), \( i \downarrow \) \( N \downarrow \)
\[ \frac{FV}{PV} = (1+i)^N \]

Divide through by \( FV \)

\[ \frac{FV}{FV} = (1+i)^N \]

Raise both side to \( \ln \) power:

\[ \left[ \frac{FV}{PV} \right]^N = \left[ (1+i) \right]^N \]

\[ t = \left[ \frac{FV}{PV} \right]^N \]

\[ c \left[ \frac{FV}{PV} \right] \]

\[ \frac{1}{c} \left[ \frac{FV}{PV} \right]^N \]

To solve for \( N \), start with:

\[ \frac{FV}{PV} = (1+i)^N \]

Take \( \ln \) of both side:

\[ \ln \left( \frac{FV}{PV} \right) = N \ln (1+i) \]

\[ \ln \left( \frac{FV}{PV} \right) \]

\[ \frac{\ln (FV/PV)}{\ln (1+i)} \]

\[ -N \]
Perpetuity

If you put $100 into a bank that pays 5% per year, you can withdraw $5 each year forever, starting one year from now. If you can withdraw the interest earnings and deposit Amount you make the deposit today so its PV = interest earnings - PV, which is a perpetual payment you can make to yourself.

\[ \text{PV} = \text{Pmt} + \text{PV} \]

which we can rewrite to be

\[ \text{PV} - \text{Pmt} \]

which is the present value of a perpetuity.

Alternate solution

\[ \text{PV} = \frac{\text{Pmt}}{1+i} + \frac{\text{Pmt}}{(1+i)^2} + \frac{\text{Pmt}}{(1+i)^3} + \ldots \]
Example: What is the PV of receiving $50/yr forever if \( i = 8\% \)?

\[
\begin{align*}
\text{PV} & \quad \text{Pmt} \quad 50 \\
\frac{1}{i} & \quad \frac{1}{0.08} \\
\end{align*}
\]

Combining the two concepts of single sum discounting and perpetuities we can compute the value of a delayed perpetuity.

Ex: What is the PV of receiving $50/yr forever if the first payment is received 6 years from now? \( i = 8\% \)

\[
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 50 & 50 & 50 & 50 \\
\hline
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & t
\end{array}
\]

\[
\begin{align*}
V_5 & \quad \text{Pmt} \quad $50 \\
\frac{1}{i} & \quad \frac{1}{0.08} \\
V_5 & \quad \frac{V_5}{(1+i)^5} \quad \frac{625}{(1.08)^5} \quad \frac{625}{1.4693} \\
V_0 & \quad \text{PV} \quad \frac{625}{1.4693} \quad \frac{625}{1.4693} \\
& \quad \frac{625}{1.4693} \quad 425.36
\end{align*}
\]
Annuity is a series of equal payments with an end date. It can be thought of as a perpetuity minus a delayed perpetuity.

\[ P \]

\[ D \]

\[ PD \]

\[ PV_P = \frac{Pmt}{i} \]

\[ PV_D = \frac{Pmt}{(1+i)^n} \]

\[ PV_{Annuity} = \frac{Pmt}{i} - \frac{Pmt}{(1+i)^n} \]

$6.75 \quad 475.36 \quad 99.64$

The present value of $50 per year for 5 years at an 8% interest rate is $1.18.$
\[
\frac{50}{1.08} + \frac{50}{(1.08)^2} + \frac{50}{(1.08)^3} + \frac{50}{(1.08)^4} + \frac{50}{(1.08)^5} = 81.92
\]

Instead of the compounding time period,

you earn 1% per month from much you have after 12 months if you start with \#1.

\[
FV = PV (1+i)^n
\]

or \( V = V (1+i) \) \((n)^2 \)

Given that you began with \#1, your interest earning over the year must be \(0.1268\).

Your % gain over the year is

\[
\frac{0.1268}{1.00} = 2.68\%\]

If you invested \#1 in a bank that pay 12% per year, after 1 year you would lose \( FV = PV (1+i)' = 1.12' = 1.12 \)
So over the year you would have earned 12.5%: 
\[ r = 2 \times \frac{12.5}{100} = 25 \% \]

Why the difference? It depends on how often interest is compounded each year.

Expressing Interest Rates on an Annual Basis

Let \( r \) be the interest rate you earn over a period.

Let \( m \) be the number of payments per year.

**APR (Annual Percentage Rate)**

\[ APR = \frac{r 	imes n}{m} \quad \text{[or } r = \frac{APR}{m} \text{]} \]

(APR is a general finance term)

However, in real estate lending (lending in general), the reported APR means something different in most cases.
Effective Annual Yield (EAY) and APR are related:

\[ \text{EAY} = \frac{100}{(1 + \frac{\text{APR}}{m})^m} - 1 \]

Where:
- \( \text{EAY} \) is the Effective Annual Yield
- \( \text{APR} \) is the Annual Percentage Rate
- \( m \) is the number of compounding periods per year

If no financing is beyond the Note rate, the FTL APR can be rounded to the nearest 1/8% or 1/16% for FTL APR to report.

Federal Reserve Board:
- APR that adjusts
- Note rate to FTL APR
- We will cover this the

Example:
- APR
- Note rate
- FTL APR
Our textbook uses the common approach to finding interest rate as \( r \% \) compounded \( y \) times \( \frac{X\%}{100} \) compounded \( y \) times in a year where \( X\% \) is the APR and \( y \) indicates \( m \).

**Example:** A bank pays \( 5\% \) compounded quarterly. What is \( r, m, \) APR and \( ENY \)?

\( APR = 5\% \)

\( m = 4 \) as there are four quarters in a year.

\( \frac{r}{m} = \frac{0.05}{4} = 0.0125 \)

\( ENY = 1 - \left(1 + \frac{r}{m}\right)^m = (1.0125)^4 - 1 \)

\( = 10.38 \%

10.38\%
Ex for a 20% note rate:

For a mortgage (with no additional fees), what is the APR?

Note: Note rates are APR's.

Legally, a promissory note, dated the term "note rate" describes the
interest rate paid on the loan.

\[ \text{EAY} = \left[ 1 + \frac{\text{APR}}{m} \right]^m \]

(Mortgages compound monthly so \( m = 12 \))

\[ \left[ 1 + \frac{12\%}{12} \right]^{12} \]

\[ - (0.01)^{12} - 1 = 0.268 \quad 2.68\% \]

Please note that a mortgage you may
charged the Note Rate/12 each month.
To be charged a 2% EAR
Will not work if your note rate is

\[ E\left(\frac{n}{2}\right) = \left(1 + r\right)^{2} - 1 \]

2. \( (1+r)^2 \)

2 \( (1+r)^2 \)

\[
\left(2 \right)^{\frac{11}{2}} \quad r
\]

\[ 1.009489 + r \]

\[ r \quad 0.009489 \quad 0.009489 \]

APR Note Rate 0.069489 \times 2

39%
Assuming no "back fees" or other charges, a loan payment is first applied to the interest that has accrued, with remainder being used to pay down the principal (i.e., to "amortize" the loan).

With a loan payment, we modify the basic equation as follows:

\[ V_{t+1} = V_t + l_t - P \text{ Payment} \]

Note: This is the balance after you make your payment.

This equation always works to determine a loan balance at any point in time, even of interest rates and payments vary.