EXAM 2: SOLUTION

Read carefully all the questions and don’t spend too much time in any single question.
Show all your work to receive credit.

1. An experimenter is convinced her measuring device had a variability measured by standard deviation \( \sigma = 2 \). During an experiment she recorded the measurements 4.1, 5.2 and 10.2. Do these data confirm or contradict her assertion? Answer by computing a 90% confidence interval for \( \sigma \). (3 points)

Answer

For these data \( n = 3, S^2 = 10.57 \). For \( \alpha = 0.1 \) we have \( \chi^2_{0.05,2} = 5.991 \) and \( \chi^2_{1-0.05,2} = 1.013 \). The 90% CI for \( \sigma^2 \) is \( \left( \frac{3 \times (10.57)}{5.991}, \frac{3 \times (10.57)}{1.013} \right) = (3.528, 205.243) \) so the 90% CI for \( \sigma \) is \( (1.8787, 14.326) \). Since this CI contains the conjectured value \( \sigma = 2 \), the data confirm the assertion.

2. It is found in a random sample of 1000 homes in a city that 228 are heated by oil. Compute a 99% confidence interval for the proportion of homes in this city that are heated by oil. (3 points)

Answer

99% CI for \( p \) is: \( \hat{p} \pm z_{0.005} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.228 \pm 2.575 \sqrt{\frac{0.228(0.772)}{1000}} = (0.194, 0.262) \).

3. Some engineering schools in the US were surveyed. The sample contained 250 electrical engineers, 80 of which were women. It also contained 175 chemical engineers, 40 of which were women. Compute a 90% confidence interval for the difference between the proportion of women in these engineering fields. Is there a significant difference between the two proportions? (3 points)

Answer

The 90% CI for \( p_1 - p_2 \) is:

\[ \hat{p}_1 - \hat{p}_2 \pm z_{0.05} \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} = 0.32 - 0.228 \pm 1.645 \sqrt{\frac{0.32(0.68)}{250} + \frac{0.228(0.772)}{175}} = (0.021, 0.163). \]

It seems to be a difference between the two proportions since the CI does not contain the values 0.

4. An assembly operation in a plant requires approximately a one-month training period for a new employee to reach maximum efficiency. A new method of training was suggested and data were collected to compare the standard and new methods. Two groups of nine employees each were trained for a period of three weeks, one group following the standard and the other the new training method. The length of time (in minutes) required for each employee to assemble the device was recorded at the end of the training period, producing the following data summaries:

Standard method: \( \bar{x} = 35.22, s_1 = 4.9441 \)
New method: \( \bar{y} = 31.56, s_2 = 4.4752 \)

Do these data present evidence to indicate that mean time to assemble at the end of the training period differs between the two training methods? Answer by computing a 95% confidence interval for the difference of means (assume the population variances are equal). (3 points)
Answer
\[ S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = \frac{8(4.9441)^2 + 8(4.4752)^2}{16} = 22.236, \text{ so } S_p = 4.715. \]
The 95\% CI for \( \mu_1 - \mu_2 \) is:
\[ \bar{x} - \bar{y} \pm t_{0.025,16} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 35.22 - 31.56 \pm 2.12(4.715) \sqrt{\frac{1}{9} + \frac{1}{9}} = (-1.052, 8.372). \]
The mean assembling times are not significantly different since the CI includes the value 0.

5. To compare the wearing qualities of two types of automobile tires, A and B say, one tire of type A and one of type B are randomly assigned and mounted on the rear wheels of each of five automobiles. The automobiles are then operated for a specified number of miles, and the amount of wear is measured for each tire, producing the following data:

<table>
<thead>
<tr>
<th>tire type</th>
<th>A</th>
<th>9.8</th>
<th>12.3</th>
<th>9.7</th>
<th>8.8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>10.2</td>
<td>9.4</td>
<td>11.8</td>
<td>9.1</td>
</tr>
<tr>
<td>difference</td>
<td>0.4</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Do these data present evidence to indicate a difference in the average wear for the two tire types? Answer by computing a 95\% confidence interval for the difference of the average wears. (3 points)

Answer
The above are paired data since the two tire types are mounted on each automobile.
For these data we have \( \bar{D} = 0.48 \) and \( S_D = 0.0837. \)
The 95\% CI for \( \mu_D = \mu_A - \mu_B \) is: \( \bar{D} \pm t_{0.025,4} \frac{S_D}{\sqrt{n}} = 0.48 \pm 2.776 \frac{0.0837}{\sqrt{5}} = (0.376, 0.584). \)
The average wears of the two tire types appear different, A being larger than B.

6. Let \( X_1, \ldots, X_n \) be a random sample from a Rayleigh distribution with probability density function
\[
 f_\theta(x) = \begin{cases} \frac{x}{\theta} e^{-\frac{x^2}{2\theta}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{with } \theta > 0.
\]

(a) Compute the method of moments estimator of \( \theta, \hat{\theta}^{MM} \)
(help: for the Rayleigh distribution it holds that \( E(X) = \frac{\sqrt{2\pi}}{\theta} \)). (2 points)

(b) Compute the maximum likelihood estimator of \( \theta, \hat{\theta}^{ML}. \) (2 points)

Answer.
(a) Solving for \( \theta \) in the equation \( \bar{X} = E(X) = \frac{\sqrt{2\pi}}{\theta} \), we get \( \hat{\theta}^{MM} = \frac{2\bar{X}^2}{\pi}. \)

(b) \( L(\theta; x) = \prod_{i=1}^{n} \frac{x_i}{\theta} e^{-\frac{x_i^2}{2\theta^2}} = \frac{1}{\theta^n} (\prod_{i=1}^{n} x_i) e^{-\frac{1}{2\theta} \sum_{i=1}^{n} x_i^2}, \) so
\[
 \ln(L(\theta; x)) = \sum_{i=1}^{n} \ln(x_i) - \frac{1}{2\theta} \sum_{i=1}^{n} x_i^2 - n \ln(\theta). \]
Solving for \( \theta \) in the equation \( 0 = \frac{d}{d\theta} \ln(L(\theta; x)) = \frac{1}{2\theta^2} \sum_{i=1}^{n} x_i^2 - \frac{n}{\theta}, \) we get \( \hat{\theta}^{ML} = \frac{1}{2n} \sum_{i=1}^{n} x_i^2. \)
7. Suppose you have a sample of size 25 from a normal distribution with mean $\mu$ and $\sigma^2 = 4$. In order to test the hypotheses $H_0 : \mu = 10$ vs $H_1 : \mu = 11.5$ you use the test that rejects $H_0$ when $\bar{X} > 11$.

(a) Compute the probability of type I error for this test (1 point)

(b) Compute the probability of type II error for this test (1 point)

(c) How large should the sample need to be in order that the test for $H_0$ versus $H_1$ has probability of type I error 0.1 and probability of type II error 0.2 (1 point)

(d) Find the test with probability of type I error 0.1 based on a random sample of size 25 (1 point)

Answer

(a) $\alpha = P(\bar{X} > 11$ when $\mu = 10) = P(Z > 2.5) = 0.0062$

(b) $\beta = P(\bar{X} \leq 11$ when $\mu = 11.5) = P(Z \leq -1.25) = 0.1056$

(c) $n = \left(\frac{2(1.28 + 0.84)}{11.5 - 10}\right)^2 = 7.99$, so $n = 8$

(d) $0.1 = P(\bar{X} > c$ when $\mu = 10) = P(Z > \frac{5(c-10)}{2})$, which implies $\frac{5(c-10)}{2} = z_{0.1} = 1.285$. Hence $c = 10.514$ and the test is: reject $H_0$ if $\bar{X} > 10.514$.

8. Determine whether the following statements are true or false:

(a) A professor is interested in the effect of sleep on students’ test performance. He chooses 50 students and gives each two tests: one given after four hours of sleep and one after eight hours of sleep. The test that should be used is the t test for paired data true (1 point)

(b) For a fixed sample size, as the probability of type II error decreases, the probability of type I error increases true (1 point)

(c) If a hypothesis test leads to incorrectly accepting the null hypothesis, then a type I error has been committed false (1 point)