The University of Texas at San Antonio
Department of Management Science and Statistics
STA 3513, Probability and Statistics
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Name: ____________________

EXAM 1: SOLUTION

1. Suppose you plan to study the respiratory physiology of the common *toado*, a small fish found in tidal pools. The pool from which you decide to collect specimens has 25 individuals, 5 of which have a fungal infection in their gills. Suppose you collect 10 fish at random from this pool.

   (a) How many outcomes there are where all fish have non-infected gills ? (2 points)

   (b) What is the probability that half of the selected fish have infected gills ? (2 points)

   (c) What is the probability that at least one fish has infected gills ? (2 points)

   Answer.

   (a) $\binom{20}{10} = 184756$

   (b) $p_b = \frac{\binom{20}{5}\binom{5}{5}}{\binom{25}{10}} = 0.00474$

   (c) $p_c = 1 - \frac{\binom{20}{10}\binom{5}{0}}{\binom{25}{10}} = 1 - 0.0565 = 0.9435$

2. A particular airline has 10 A.M. flights from Chicago to New York, Atlanta and Los Angeles. Let A denote the event that the New York flight is full, and define events B and C analogously for the flights to Atlanta and Los Angeles. Suppose $P(A) = 0.6$, $P(B) = 0.5$, $P(C) = 0.4$ and the three events are independent. Compute the probability that

   (a) All three flights are full. (2 points)

   (b) At least one flight is not full. (2 points)

   (c) Exactly one of the three flights is full. (2 points)

   Answer.

   (a) $p_a = P(A \cap B \cap C) = (0.6)(0.5)(0.4) = 0.12$

   (b) $p_b = 1 - p_b = 1 - 0.12 = 0.88$

   (c) $p_c = P(A \cap B' \cap C') + P(A' \cap B \cap C') + P(A' \cap B' \cap C)$
   $= (0.6)(0.5)(0.6) + (0.4)(0.5)(0.6) + (0.4)(0.5)(0.4) = 0.38$

3. A certain company sends 40% of its overnight mail parcels via express mail service E_1, 50% via express mail service E_2 and the remaining 10% are sent via E_3. Of those sent by E_1, 2% arrive after the guaranteed delivery time (called a "late" delivery and denoted by L); of those sent by E_2, 1% arrive late, whereas 5% of the parcels sent by E_3 arrive late.

   (a) What is the probability that a randomly selected parcel arrives late ? (2 points)
   (b) If a randomly selected parcel has arrived on time, what is the probability that it was not sent via E_1? (2 points)

   Answer.

   (a) $P(L) = P(L \mid E_1)P(E_1) + P(L \mid E_2)P(E_2) + P(L \mid E_3)P(E_3)$
   $= (0.02)(0.4) + (0.01)(0.5) + (0.05)(0.1) = 0.018$
   (b) $P(E_1' \mid L') = 1 - P(E_1 \mid L') = 1 - \frac{P(L' \mid E_1)P(E_1)}{P(L')} = 1 - \frac{(0.98)(0.4)}{0.982} = 0.6008$

4. Two fair six-sided dice are tossed independently, and let M be the maximum of the two tosses.

   (a) Compute the probability mass function (pmf) of the random variable M. (2 points)
   (b) Compute the cumulative distribution function (cdf) of the random variable M. (2 points)
   (c) Compute $P(3 < M < 6)$. (2 points)

   Answer.

   (a) $R_M = \{1, 2, 3, 4, 5, 6\}$.
   $f(1) = P(M = 1) = P((1, 1)) = 1/36$, $f(2) = P(M = 2) = P((1, 2), (2, 1), (2, 2)) = 3/36$,
   $f(3) = 5/36$, $f(4) = 7/36$, $f(5) = 9/36$, $f(6) = 11/36$.
   (b)

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1/36 & \text{if } 1 \le x < 2 \\ 4/36 & \text{if } 2 \le x < 3 \\ 9/36 & \text{if } 3 \le x < 4 \\ 16/36 & \text{if } 4 \le x < 5 \\ 25/36 & \text{if } 5 \le x < 6 \\ 1 & \text{if } x \ge 6 \end{cases}.$$

   (c) $P(3 < M < 6) = f(4) + f(5) = 16/36$

5. An important factor in solid missile fuel is the particle size distribution. Significant problems occur if the particle sizes are too large. From production data in the past, it has been determined that the particle size (in micrometers) is a continuous random variable with probability density function (pdf)

$$f(x) = \begin{cases} 3x^{-4} & \text{if } x > 1 \\ 0 & \text{if } x \le 1 \end{cases}$$

   (a) Verify that this is a valid pdf. (2 points)
   (b) Find $F(x)$, the cumulative distribution function corresponding to the above pdf. (2 points)
   (c) What is the probability that a particle chosen at random from the manufactured fuel exceeds 4 micrometers ? (2 points)

   Answer.

   (a) It holds that $f(x) \ge 0$ for all $x \in R$, and $\int_{-\infty}^{\infty} f(x)dx = \int_1^{\infty} 3x^{-4}dx = 1$.
   (b) $F(x) = \int_{-\infty}^{x} f(y)dy$. Then, for $x \le 1$, $F(x) = 0$, while for $x > 1$, $F(x) = \int_1^x 3y^{-4}dy = 1 - x^{-3}$.
   (c) $P(X > 4) = 1 - P(X \le 4) = 1 - F(4) = 1 - (1 - 4^{-3}) = 0.015$.