WARP and combinatorial choice*

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Abstract

For combinatorial choice problems, I show that the *Irrelevance of Rejected Items* condition is equivalent to the *Weak Axiom of Revealed Preference* (WARP), and is necessary and sufficient for the existence of a complete, reflexive and antisymmetric rationalization of a combinatorial choice function. I also show the equivalence of WARP to path independence and to other classical choice conditions when the choice domain is combinatorial.

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1 Introduction

In combinatorial choice problems (Echenique, 2007), an economic actor chooses out of a set of available *items* Y some subset of them $C(Y)$. A *combinatorial choice function* C describes the choice of this actor at every set Y drawn from a universal set X.

Combinatorial choice problems naturally arise in matching markets and markets with indivisible goods. In the matching of workers to firms in a labor market, for instance, each firm hires a team of workers to fill a set of positions (Kelso and Crawford, 1982; Roth, 1984). The decision of whether to hire a particular worker in general depends upon which other workers are being hired. Hence, the firm's choice to assemble a particular team from a pool of workers is in fact a choice of this team in favor of every other possible team. In centralized school assignment, an indivisible goods

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allocation problem, over-subscription at a school requires a decision about which students to turn down. Policy objectives such as diversity imply that this decision depends upon which other students are available (Sönmez and Switzer, 2013; Echenique and Yenmez, 2015; Kominers and Sönmez, 2016). Since such objectives involve comparisons of groups of students rather than individuals, the decision of which students to accept from the set of applicants is a combinatorial choice problem.

In this paper, I analyze the rationality properties of combinatorial choice functions. A combinatorial choice function is rationalizable if chosen sets are necessarily the best ones amongst all sets constructible from a given set of available items, as determined by binary relation over sets of items called the rationalizing preference. The main result (Theorem 1) is a series of novel characterizations of rationalizable combinatorial choice functions by various conditions on choice.

The investigation of rationality of choice from *budget sets* of *mutually exclusive* alternatives has a rich tradition dating back to the analysis of demand through by way of the Weak Axiom of Revealed Preference (WARP) (Samuelson, 1938). Say an actor reveals a preference for one alternative over another if it ever chooses the first when the second is available. WARP is the requirement that in this case the actor cannot also reveal a preference for second over the first.[1] I analyze combinatorial choice in the spirit of this revealed preference approach.

Combinatorial choice, however, differs from the classical approach by describing choice from *opportunity sets* of *combinable* items. The flexibility to combine items from an opportunity set into a desired consumption *bundle* implies that the actual budget set of mutually exclusive alternatives is the set of all bundles that could be constructed from items in the opportunity set—budget sets are combinatorially generated from opportunity sets. To illustrate, suppose a firm has a worker pool $Y = \{w_1, w_2\}$. The firm could choose the team of two ($\{w_1, w_2\}$) or a team of just one worker ($\{w_1\}$ or $\{w_2\}$). It could also turn them all down, i.e. it could choose the empty set ($\varnothing$). Then the budget set is the collection of these four teams as mutually exclusive alternatives. Therefore, to understand the implication of WARP or any other axiom in the classical approach, I introduce the following novel representation of a combinatorial choice function C. For each opportunity set Y, combinatorially generate the associated budget set and map this budget set to the bundle $C(Y)$. This uniquely describes a choice function in the classical sense of selecting an alternative from a set of mutually exclusive alternatives. I call this classical choice function the *faithful representation* of C, and use it to study WARP and other classical choice axioms.

Theorem 1 states the equivalence of a variety of choice conditions with the rationalizability of a combinatorial choice function. There are two characterizing conditions of note. The first condition WARP. The second condition is the Irrelevance of Rejected Items (IRI), defined for combinatorial choice.[2] Say an actor rejects an item from an opportunity set if it is not part of the chosen bundle.

[1] There are a few variations of WARP (Samuelson, 1938; Arrow, 1959). See Section 2.4 for the formal definition I use in this paper.

[2] Aygün and Sönmez (2013) introduce this condition for the matching with contracts model (Hatfield and Milgrom,

IRI is the requirement that the actor's chosen bundle does not change when the opportunity set is shrunk by the removal of the rejected item. I show that a combinatorial choice function satisfies IRI if and only if its faithful representation satisfies WARP, and moreover each of these is necessary and sufficient for the existence of a rationalizing preference. Furthermore, requiring the rationalizing preference to be complete, reflexive and antisymmetric is without any loss of generality.

One contribution of Theorem 1 is to matching theory. IRI is a crucial requirement for many important results in choice-based matching (Aygün and Sönmez, 2013). However, it is only defined for combinatorial choice since it directly references items. On the other hand, WARP is a widely studied condition definable in any model of choice. Theorem 1 shows that IRI is in fact an incarnation of WARP in the combinatorial setting and is exactly the combinatorial choice condition to ensure rationalizability. Furthermore, I show that a suitable variant of IRI for a combinatorial choice correspondence is equivalent to WARP for its faithful representation (Proposition 1). These results provide a rational choice foundation for the use of IRI in matching models.

The domain of a classical choice function is combinatorial if the function is the faithful representation of some combinatorial choice function. This class of domains has not been previously analyzed, to the best of my knowledge. Theorem 1 also contributes to the choice theory literature by demonstrating the equivalence of some widely studied classical choice axioms to WARP when the domain of a classical choice function is combinatorial.[3] Relations amongst these axioms have been studied for other domains, notably the various domains with finite budget sets (Sen, 1971; Bossert et al., 2006).

Finally, note that a preference relation that rationalizes a combinatorial choice function need not be transitive. A sufficient condition on combinatorial choice functions to obtain transitivity of the rationalizing preference is substitutability (Proposition 2). This condition states that a chosen item is still chosen if the opportunity set is shrunk by the removal of one or more other items. Whether there is a simple combinatorial choice condition that is both necessary and sufficient for the existence of some transitive and rationalizing relation is an open question.

Related Literature Interpreting items as alternatives and opportunity sets as budget sets yields another representation of a combinatorial choice function as a classical choice function. Because the mutual exclusivity requirement is lost, I call it the *forgetful representation*. Generally, the implications of a choice axiom for a combinatorial choice function will depend on whether it is imposed on the faithful or the forgetful representation. The forgetful representation has been used in the prior matching literature. For instance, Chambers and Yenmez (2017) study combinatorial choice

2005) as the Irrelevance of Rejected Contracts. Versions of this condition appear earlier in the matching literature as well (Blair, 1988; Alkan, 2002; Fleiner, 2003).

[3] The axioms are path independence, the Chernoff property, Arrow's axiom, and the independence of irrelevant alternatives.

functions using the forgetful representation implicitly. They note that the combination of IRI and substitutability is equivalent to path independence[4] of the forgetful representation. They use this observation together with a result on path independent choice (Aizerman and Malishevski, 1981) to provide a novel analysis of many-to-many matching problems. In contrast, I show that path independence of the faithful representation of a combinatorial choice function is equivalent only to IRI (Theorem 1). So, path independence places fewer restrictions on combinatorial choice if it is demanded of the faithful rather than the forgetful representation.

By interpreting alternatives as items and budget sets as opportunity sets, a classical choice correspondence can be mapped to combinatorial choice function in a manner that is the inverse of the forgetful representation. Brandt and Harrenstein (2011) study set-rationalizability, which is rationalization of a classical choice correspondence by a binary relation on the *power* set of alternatives. One of their main results is that a condition they call $\hat{\alpha}$ characterizes classical choice rules that are set-rationalizable (their Theorem 2). It is straightforward to note rationalizability and IRI of a combinatorial choice function implies set-rationalizability and condition $\hat{\alpha}$ of the forgetful representation. Since every classical choice correspondence is the forgetful representation of a combinatorial choice function, Theorem 1 produces the same characterization as their Theorem 2. However, the interpretations of these two results are different, and the other results in our papers diverge.

Finally, Chambers and Echenique (Forthcoming) characterize combinatorial *demand* functions that are rationalizable by quasilinear preferences by the properties of continuity and the law of demand. There is a homogeneous and continuous good, money, in addition to a set of items and the bundles chosen are a functions of a money price for each item. Their model is suitable for the analysis of package auctions and competitive markets with discrete goods and money. On the other hand, in my model goods are completely heterogeneous and budget sets are combinatorially determined by a set of available items rather than by a price vector, making it the appropriate framework for the study of markets without transferable utility. In this sense, our studies are complementary.

The rest of the paper is organized as follow. In Section 2, I define the combinatorial and classical choice models, the faithful and forgetful representations, and a variety of choice conditions. In Section 3, I describe the main results of the paper. In Section 4, I provide proofs of all the results.

[4] Path independence requires that the choice from a union of sets is the union of the choices from each of the sets.

2 Two choice models

2.1 Combinatorial choice

A *combinatorial choice model* is a pair (X, C) consisting of a nonempty set X and a correspondence $C : 2^X \rightrightarrows 2^X$. Elements of X are *items*. A *bundle* is a subset of items Z being consumed, where Z could be the empty set $\varnothing$. An *opportunity set* is a subset of items Y that determines which items are available for the construction of bundles. Correspondence C is the *combinatorial choice rule* that maps every opportunity set $Y \in 2^X$ to a nonempty collection of bundles $C(Y)$ such that every chosen bundle $Z \in C(Y)$ is constructed from available items ($Z \subseteq Y$).[5]

A combinatorial choice rule C is *single-valued* if $|C(Y)| = 1$, and *multi-valued* otherwise. Note that C being single-valued implies that for every $Y \subseteq X$, $C(Y) = \{Z\}$ for some $Z \in 2^Y$. So, it is possible that $|C(Y)| = 1$ and $|Z| > 1$, because the unique bundle chosen is composed of multiple items. If single-valued, refer to C as a *combinatorial choice function* and identify $C(Y)$ with the unique bundle it contains. In this case, $C(Y) = \varnothing$ means that the choice from Y is the empty bundle.

2.2 Classical choice

A *classical choice model* is a triple $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ consisting of nonempty sets $\mathcal{X}$ and $\mathscr{B}$ and a correspondence $\mathbf{c} : \mathscr{B} \rightrightarrows \mathcal{X}$. Elements of $\mathcal{X}$ are *mutually exclusive alternatives*. Subsets $\mathcal{B} \subseteq \mathcal{X}$ are *budget sets*.[6] The *choice domain* $\mathscr{B}$ is a nonempty collection of budget sets, i.e. $\mathscr{B} \subseteq 2^{\mathcal{X}}$. Correspondence $\mathbf{c}$ is the *choice rule*, where for every budget set in the choice domain ($\mathcal{B} \in \mathscr{B}$), every chosen alternative must be available ($\mathbf{c}(\mathcal{B}) \subseteq \mathcal{B}$) and at least one alternative must be chosen if at least one is available ($\mathcal{B} \neq \varnothing$ implies $\mathbf{c}(\mathcal{B}) \neq \varnothing$).

A choice rule $\mathbf{c}$ is *single-valued* if, for all nonempty $\mathcal{B} \in \mathscr{B}$, exactly one element of $\mathcal{B}$ is chosen: $|\mathbf{c}(\mathcal{B})| = 1$. It is *multi-valued* otherwise. When single-valued, the choice rule $\mathbf{c}$ can be naturally identified as a *choice function* $\mathbf{c} : \mathscr{B} \to \mathcal{X}$. A choice function $\tilde{\mathbf{c}} : \mathscr{B} \to \mathcal{X}$ is a *selection* from a choice rule $\mathbf{c}$ if, for all nonempty $\mathcal{B} \in \mathscr{B}$, $\tilde{\mathbf{c}}(\mathcal{B}) \in \mathbf{c}(\mathcal{B})$.

2.3 Representing combinatorial choice

Axioms for the classical choice model rely upon the interpretation that elements in $\mathcal{X}$ are mutually exclusive alternatives. The following representation of a combinatorial choice model as a classical one is faithful to this mutual exclusivity requirement.

[5] It is possible that $C(Y)$ contains only the empty bundle $\varnothing$ for some $Y \in 2^X$, i.e. $C(Y) = \{\varnothing\}$, but $C(Y)$ being the empty collection is not allowed.

[6] Allowing for an empty budget set is useful to show how a classical choice model is a particular kind of representation of a combinatorial choice model, and is harmless.

Definition 1. *The faithful representation of a combinatorial choice model* (X, C) *is the classical choice model* $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ *defined by*

1. $\mathcal{X} = 2^X$,
2. $\mathscr{B} = 2^{2^X}$,
3. $\mathbf{c}(2^Y) = C(Y)$ *for every* $Y \subseteq X$.

A commonly used alternative representation is the forgetful representation, where items are identified as alternatives and opportunity sets as budget sets. This representation "forgets" the mutual exclusivity requirement for elements in $\mathcal{X}$. Any condition applied to the forgetful representation has the same implications as applying the syntactically identical condition to the original combinatorial choice model. However, since the interpretation of most axioms rely upon the interpretation that elements in $\mathcal{X}$ are mutually exclusive, the forgetful representation does not preserve semantic content.

Definition 2. *The forgetful representation of* (X, C) *is the classical choice model* $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ *defined by*

1. $\mathcal{X} = X$,
2. $\mathscr{B} = 2^X$,
3. $\mathbf{c}(Y) = C(Y)$ *for every* $Y \subseteq X$.

Note that if the combinatorial choice rule is not just single-valued but also the bundles chosen contain at most one item (unit-demand), then items are essentially alternatives. In this case, the combinatorial choice model is the same as a classical choice model with a complete choice domain, i.e. the forgetful and faithful representations coincide.

2.4 Choice conditions, rationalizability, and revealed preference

Binary relations A *binary relation* R on a set A is a subset of the product space $A \times A$. For every $a, a' \in A$, let $a\ R\ a'$ denote $(a, a') \in R$. A binary relation R on a set A is *reflexive* if $a\ R\ a$ for every $a \in A$. It is *complete* if for every $a, a' \in A$ with $a \neq a'$, $a\ R\ a'$ or $a'\ R\ a$. It is *antisymmetric* if for every $a, a' \in A$, if $a\ R\ a'$ and $a'\ R\ a$ then $a = a'$. It is *asymmetric* if for every $a, a' \in A$, $a\ R\ a'$ implies $\neg(a'\ R\ a)$. It is *transitive* if for every $a, a', a'' \in A$, $a\ R\ a'$ and $a'\ R\ a''$ implies $a\ R\ a''$. It is a *linear order* if it is complete, reflexive, transitive, and antisymmetric. A binary relation R on A is an *extension* of a binary relation R' on A if for every $a, a' \in A$, *i)* $a\ R'\ a'$ implies $a\ R\ a'$ *ii)* $a\ R'\ a'$ and $\neg(a\ R'\ a')$ implies $a\ R\ a'$ and $\neg(a\ R\ a')$ The *transitive closure* $\bar{R}$ of a binary relation R on set A is the smallest transitive relation that contains R. Formally, $\bar{R} = \bigcap_{R' \in \mathcal{R}^t} R'$, where $\mathcal{R}^t$ is the set of all transitive relations on A.

Combinatorial choice conditions, rationalizability, the Blair relation

Let (X, C) be a combinatorial choice model with single-valued C. Combinatorial choice function C satisfies *irrelevance of rejected items (IRI)* (Alkan, 2002; Aygün and Sönmez, 2013) if the following holds: for every $Y, Y' \subseteq X$, if $C(Y) \subseteq Y' \subseteq Y$, then $C(Y) = C(Y')$. A combinatorial choice function C satisfies *substitutability* (Kelso and Crawford, 1982; Roth, 1984) if the following holds: for every $Y, Y' \subseteq X$ and any $y \in Y$, if $Y \subseteq Y'$ and $y \in C(Y')$, then $y \in C(Y)$. Equivalently, $C(Y') \cap Y \subseteq C(Y)$.

Given a set of items X and a binary relation R on 2^X, define the correspondence $C^R : 2^X \rightrightarrows 2^X$ as follows: for every $Y \subseteq X$, $C^R(Y) = \{Z \subseteq Y : \text{for all } Z' \subseteq Y, Z \ R \ Z'\}$, i.e. $C^R(Y)$ is the set of R-greatest bundles amongst those that can be constructed from opportunity set Y. Given some combinatorial choice model (X, C), say C is *rationalizable* if there exists a binary relation R on 2^X such that $C = C^R$. If R is transitive, then C is *transitively rationalizable*.

For a combinatorial choice function C, the *Blair relation* R^B_C is a binary relation defined on 2^X as follows: $Z \ R^B_C \ Z'$ if and only if $Z = C(Z \cup Z')$, where $Z, Z' \in 2^X$ (Blair, 1988). Note that R^B_C is an antisymmetric relation since C is a function.

Classical choice conditions, rationalizability, and revealed preference

Let $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ be a classical choice model. Choice domain $\mathscr{B}$ is *complete* if $\mathscr{B} = 2^{\mathcal{X}}$. Choice domain $\mathscr{B}$ is *combinatorial* if $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ is the faithful representation of some combinatorial choice model (X, C).

Choice rule $\mathbf{c}$ satisfies *independence of irrelevant alternatives (IIA)* (Nash, 1950) if the following holds: For every $\mathcal{B}, \mathcal{B}' \in \mathscr{B}$, if $\mathbf{c}(\mathcal{B}') \subseteq \mathcal{B} \subseteq \mathcal{B}'$, then $\mathbf{c}(\mathcal{B}) = \mathbf{c}(\mathcal{B}')$. Choice rule $\mathbf{c}$ has the *Chernoff property* (Chernoff, 1954), also called condition α (Sen, 1971), if the following holds: for every $\mathcal{B}, \mathcal{B}' \in \mathscr{B}$ and for every $Z \in \mathcal{B}$, if $\mathcal{B} \subseteq \mathcal{B}'$ and $Z \in \mathbf{c}(\mathcal{B}')$, then $Z \in \mathbf{c}(\mathcal{B})$. Choice rule $\mathbf{c}$ satisfies *Arrow's axiom* (Arrow, 1959) if the following holds: for all $\mathcal{B}, \mathcal{B}' \in \mathscr{B}$, if $\mathcal{B} \subseteq \mathcal{B}'$ and $\mathbf{c}(\mathcal{B}') \cap \mathcal{B} \neq \emptyset$, then $\mathbf{c}(\mathcal{B}) = \mathbf{c}(\mathcal{B}') \cap \mathcal{B}$. For a classical choice model $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ with *complete* domain $\mathscr{B}$, choice rule $\mathbf{c}$ satisfies *Plott path independence* (Plott, 1973) if the following holds: for every $\mathcal{B}, \mathcal{B}' \in \mathscr{B}$, $\mathbf{c}(\mathcal{B} \cup \mathcal{B}') = \mathbf{c}(\mathbf{c}(\mathcal{B}) \cup \mathcal{B}')$. I extend Plott path independence to models with arbitrary domains: choice rule $\mathbf{c}$ satisfies *path independence* if for every $\mathcal{B}, \mathcal{B}' \in \mathscr{B}$ such that $\mathcal{B} \cup \mathcal{B}' \in \mathscr{B}$, $\mathbf{c}(\mathcal{B} \cup \mathcal{B}') = \mathbf{c}(\mathbf{c}(\mathcal{B}) \cup \mathcal{B}')$ whenever $\mathbf{c}(\mathcal{B}) \cup \mathcal{B}' \in \mathscr{B}$.

Given a set of alternatives $\mathcal{X}$, a choice domain $\mathscr{B}$, and a binary relation R on $\mathcal{X}$, the *choice rule generated by* R, denoted $\mathbf{c}^R$, is defined as follows: for every $\mathcal{B} \in \mathscr{B}$, $\mathbf{c}^R(\mathcal{B}) = \{Y \in \mathcal{B} : \text{for all } Z \in \mathcal{B}, Y \ R \ Z\}$, i.e. $\mathbf{c}^R(\mathcal{B})$ is the set of R-greatest elements amongst the alternatives in the budget set $\mathcal{B}$. For $(\mathcal{X}, \mathscr{B}, \mathbf{c})$, say $\mathbf{c}$ is *rationalizable* if there exists a binary relation R on $\mathcal{X}$ such that $\mathbf{c} = \mathbf{c}^R$. If R is transitive, then $\mathbf{c}$ is *transitively rationalizable*.

Define the *revealed preference relation* $R_{\mathbf{c}}$ of a classical choice model $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ as follows: for all $Z, Z' \in \mathcal{X}$, $Z \ R_{\mathbf{c}} \ Z'$ if and only if there exists $\mathcal{B} \in \mathscr{B}$ such that $Z \in \mathbf{c}(\mathcal{B})$ and $Z' \in \mathcal{B}$. Say that Z is revealed preferred to Z' if $Z \ R_{\mathbf{c}} \ Z'$. Define the *revealed strict preference relation* $R^s_{\mathbf{c}}$ as follows: for all $Z, Z' \in \mathcal{X}$, $Z \ R^s_{\mathbf{c}} \ Z'$ if and only if there exists $\mathcal{B} \in \mathscr{B}$ such that $Z \in \mathbf{c}(\mathcal{B})$ and $Z' \in \mathcal{B} \setminus \mathbf{c}(\mathcal{B})$. Say that Z is revealed strictly preferred to Z' if $Z \ R^s_{\mathbf{c}} \ Z'$. A classical choice model $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ satisfies the *weak axiom of revealed preference (WARP)* if for every $Z, Z' \in \mathcal{X}$, $Z \ R^s_{\mathbf{c}} \ Z'$ implies $\neg (Z' \ R_{\mathbf{c}} \ Z)$. It satisfies the *strong axiom of revealed preference (SARP)* if for every $Z, Z' \in \mathcal{X}$, $Z \ \bar{\mathcal{R}}^s_{\mathbf{c}} \ Z'$ implies $\neg (Z' \ R_{\mathbf{c}} \ Z)$, where $\bar{R}^s_{\mathbf{c}}$ is the transitive closure of $R^s_{\mathbf{c}}$.

3 Results

The main result of the paper is the following theorem. It states that any one of a variety of choice conditions is necessary and sufficient to obtain a rationalizing preference.

Theorem 1. *Let (X, C) be a single-valued combinatorial choice model, and let $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ be its faithful representation. Then the following statements are equivalent.*

1. *C satisfies IRI.*
2. *$\mathbf{c}$ satisfies WARP.*
3. *$\mathbf{c}$ satisfies IIA.*
4. *$\mathbf{c}$ satisfies path independence.*
5. *$\mathbf{c}$ satisfies the Chernoff property.*
6. *$\mathbf{c}$ satisfies Arrow's axiom.*
7. *The Blair relation R^B_C on 2^X is order isomorphic to the revealed preference relation $R_{\mathbf{c}}$ on $\mathcal{X}$.*
8. *The Blair relation R^B_C rationalizes C.*
9. *There exists a complete, reflexive, and antisymmetric binary relation on 2^X that rationalizes C.*
10. *There exists a binary relation on 2^X that rationalizes C.*

Alkan (2002) observes that the Blair relation resembles a revealed preference relation.[7] Theorem 1 formalizes this idea, demonstrating their equivalence when the combinatorial choice function satisfy IRI. However, this may not be true without IRI.[8] It is also notable that completeness and antisymmetry of preferences have no testable implications given rational choice.

[7] The Blair relation is useful in understanding the core and the stable set in matching problems. In the context of classical many-to-one matching, Martínez et al. (2012) use the Blair relation to identify the information in preferences that is relevant to determine the set of core matches. Echenique and Oviedo (2006) use the Blair relation to study a version of the core in the many-to-many setting.

[8] Define (X, C) by $X = \{x, y\}$ and $C(Y) = Y$ if $Y \subseteq X \setminus \{y\}$ and $C(Y) = \emptyset$ if $y \in Y$. Then C satisfies substitutability but not IRI. Note that $C(X) = \emptyset$ implies $\emptyset \ R_{\mathbf{c}} \ \{x\}$, where $\mathbf{c}$ is the faithful representation of C. However, $\emptyset \subseteq \{x\}$ and $C(\{x\}) = \{x\}$, so $\{x\} \ R^B_C \ \emptyset$. Since R^B_C is antisymmetric by definition, $\neg(\emptyset \ R^B_C \ \{x\})$.

Theorem 1 has novel implications for classical choice theory, because it provides results for classical choice functions on combinatorial domains. WARP is not necessary for rationalizability of a choice rule when the choice domain is arbitrary (Richter, 1971), even when the choice rule is single-valued. However, if the choice domain contains every doubleton and tripleton of alternatives as budget sets, WARP is necessary and sufficient for transitive rationalizability (Sen, 1971). Theorem 1 shows the novel result that WARP characterizes rationalizability, albeit without transitivity, on the combinatorial choice domain, even though this domain does not satisfy Sen's doubleton and tripleton requirement. The theorem also obtains new equivalences for other conditions on classical choice functions with combinatorial domains.

I now consider combinatorial choice rules, i.e multi-valued combinatorial choice. For instance, priorities in the Boston and New York City public school systems have ties. A lottery is used to break ties and obtain a strict ordering over students . The combinatorial choice rules generated from priority structures with ties will generally be multi-valued.[9] Since tie-breakers are a common approach to make choice single-valued in applications (Erdil and Ergin, 2008; Abdulkadiroğlu et al., 2009), a natural extension of IRI to arbitrary combinatorial rules is to insist that standard IRI be satisfied by every selection obtained via an exogenous tie-breaker. Proposition 1 below offers a foundation for this natural extension. It partly extends Theorem 1 by showing IRI of a combinatorial choice rule is satisfied if and only if WARP is satisfied by its faithful representation.

Let (X, C) be a combinatorial choice model. A *tie-breaker* τ for (X, C) is a complete, transitive, and asymmetric binary relation on 2^X. Let C^τ denote the selection from C defined as follows: for all $Y \subseteq X$, $C^\tau(Y) = Z$, where $Z \in C(Y)$, $Z \;\tau\; Z'$ for all $Z' \in C(Y) \setminus \{Z\}$. In words, C^τ is a tie-broken choice function from C. Combinatorial choice rule C satisfies *extended-IRI* if and only if for every tie-breaker τ for (X, C), C^τ satisfies IRI. This natural extension requires that the systematic approach of resolving indecisiveness through tie-breaking guarantees that the realized single-valued choice satisfies IRI. Of course, extended-IRI is equivalent to IRI when C is single-valued. It is important that the single-valued selection is obtained using a tie-breaker. Proposition 1 does not hold if arbitrary selections are allowed.[10]

Proposition 1. *Let (X, C) be a combinatorial choice model, and let $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ be its faithful representation. Then $\mathbf{c}$ satisfies WARP if and only if C satisfies extended-IRI.*

While Proposition 1 is a partial analogue of Theorem 1 for multi-valued combinatorial choice, a full analogue cannot be obtained. For instance, WARP implies IIA in any classical choice model

[9] See Erdil and Kumano (2014) and Alva and Manjunath (2017) for recent papers that work directly with multi-valued combinatorial choice rules.

[10] Consider the following counterexample: Let $X = \{x, y, z\}$, with every subset a feasible bundle. Define C by $C(Y) = 2^Y$ for $Y \subseteq X$, whose faithful representation $\mathbf{c}$ satisfies WARP trivially. However, consider the selection, $\tilde{C}$ satisfying: for each $Y \subseteq X$, $\tilde{C}(Y) = \{x\}$ if $x \in Y$ and $z \notin Y$, and $\tilde{C}(Y) = \{y\}$ if $y \in Y$ and $z \in Y$. Then, $\tilde{C}$ violates IRI and its faithful representation $\tilde{\mathbf{c}}$ violates WARP, since $\tilde{C}(\{x, y\}) = \{x\}$ and $\tilde{C}(\{x, y, z\}) = \{y\}$.

(Lemma 1 in the Appendix). However, Example 1 below reveals that even for combinatorial domains, the equivalence between these properties relies upon choice being single-valued.[11] Therefore, a combinatorial choice model need not satisfy extended-IRI even if its faithful representation satisfies IIA. The same can be stated about the Chernoff property or Arrow's axiom in place of IIA.

Example 1. Let (X, C) be the combinatorial choice model defined by $X = \{x, y, z\}$ and C satisfying:

Y	$C(Y)$	Y	$C(Y)$
$\emptyset$	$\{\emptyset\}$	$\{x, y\}$	$\{\{x\}\}$
$\{x\}$	$\{\{x\}\}$	$\{x, z\}$	$\{\{x\}\}$
$\{y\}$	$\{\{y\}\}$	$\{y, z\}$	$\{\{y\}, \{z\}\}$
$\{z\}$	$\{\{z\}\}$	$\{x, y, z\}$	$\{\{y\}, \{z\}\}$

Let $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ be its faithful representation. It can be shown that $\mathbf{c}$ satisfies IIA. However, it does not satisfy WARP, since $\{x\}\ R^s_{\mathbf{c}}\ \{y\}$ (via budget set $2^{\{x,y\}}$) but $\{y\}\ R^s_{\mathbf{c}}\ \{x\}$ (via budget set $2^{\{x,y,z\}}$). Nor is it rationalizable. Moreover, it can be verified that *every* selection from C will violate IRI (and every selection from $\mathbf{c}$ will violate WARP).

Finally, I consider the possibility of transitive rationalizability. Some of the results in the following proposition are not new, but I state the equivalence here to contrast it with Theorem 1.

Proposition 2. *Let (X, C) be a single-valued combinatorial choice model, and let $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ be its faithful representation. If C satisfies substitutability,*[12] *then the following statements are equivalent.*

1. $\mathbf{c}$ *satisfies WARP.*
2. $\mathbf{c}$ *satisfies SARP.*
3. R^B_C *transitively rationalizes C.*
4. *There exists a linear order on 2^X that rationalizes C.*

Weaker versions of substitutability have been studied in the matching with contracts literature (Hatfield and Kojima, 2010). However, Proposition 2 is tight in the sense that these weaker versions would not suffice. Aygün and Sönmez (2012), for instance, have examples satisfying IRI and weakened substitutability that do not satisfy SARP.

[11] The equivalence can also fail when choice is single-valued if the domain is not combinatorial. Consider $\mathscr{B} = \{\mathcal{B}, \mathcal{B}'\}$, where $\mathcal{B} = \{Z_1, Z_2, Z_3\}$ and $\mathcal{B}' = \{Z_1, Z_2, Z_4\}$, and for $i, j \in \{1, 2, 3, 4\}$, $Z_i \in \mathcal{X}$ for some set of mutually exclusive alternatives $\mathcal{X}$ and $Z_i = Z_j$ if and only if $i = j$. Suppose $\mathbf{c}(\mathcal{B}) = \{Z_1\}$ and $\mathbf{c}(\mathcal{B}') = \{Z_2\}$. Then choice is single-valued, and WARP is violated, but IIA is trivially satisfied. Moreover, $\mathbf{c}$ is rationalizable by the revealed preference relation $R_{\mathbf{c}}$.

[12] The equivalence to statements 2. - 4. holds without substitutability.

4 Proofs

The following two lemmas are useful to prove the main equivalence theorem, Theorem 1.

Lemma 1. *Suppose a classical choice model* $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ *satisfies WARP. Then it satisfies IIA.*

Proof of Lemma 1. I begin with the following claim.

Claim: If $Y \ R_{\mathbf{c}} \ Z$ and $Z \ R_{\mathbf{c}} \ Y$, then for every $\mathcal{B} \in \mathscr{B}$ such that $Y, Z \in \mathcal{B}$, $Y \in \mathbf{c}(\mathcal{B})$ if and only if $Z \in \mathbf{c}(\mathcal{B})$.

Proof of Claim: By WARP, $\neg(Y \ R^s_{\mathbf{c}} \ Z)$ and $\neg(Z \ R^s_{\mathbf{c}} \ Y)$. Given that $Y \in \mathbf{c}(\mathcal{B})$ and $Z \in \mathcal{B}$, $\neg(Y \ R^s_{\mathbf{c}} \ Z)$ implies $Z \in \mathbf{c}(\mathcal{B})$. Symmetrically, it can be established that $Y \in \mathbf{c}(\mathcal{B})$ if $Z \in \mathbf{c}(\mathcal{B})$ and $Y \in \mathcal{B}$. ■

Let $\mathcal{B}, \mathcal{B}' \in \mathscr{B}$, $\mathcal{B}' \subseteq \mathcal{B}$ and suppose $\mathbf{c}(\mathcal{B}) \subseteq \mathcal{B}'$. By definition, $\mathbf{c}(\mathcal{B}') \subseteq \mathcal{B}'$. I need to show that $\mathbf{c}(\mathcal{B}') = \mathbf{c}(\mathcal{B})$.

Let $Y, Z \in \mathbf{c}(\mathcal{B})$. By the claim and the hypothesis that $\mathbf{c}(\mathcal{B}) \subseteq \mathcal{B}'$, $Y \in \mathbf{c}(\mathcal{B}')$ if and only if $Z \in \mathbf{c}(\mathcal{B}')$. Thus, if $\mathbf{c}(\mathcal{B}) \cap \mathbf{c}(\mathcal{B}') \neq \emptyset$, then $\mathbf{c}(\mathcal{B}) \subseteq \mathbf{c}(\mathcal{B}')$.

Now, let $Y, Z \in \mathbf{c}(\mathcal{B}')$. By definition and by hypothesis, $\mathbf{c}(\mathcal{B}') \subseteq \mathcal{B}' \subseteq \mathcal{B}$, so by the claim, $Y \in \mathbf{c}(\mathcal{B})$ if and only if $Z \in \mathbf{c}(\mathcal{B})$. Thus, if $\mathbf{c}(\mathcal{B}) \cap \mathbf{c}(\mathcal{B}') \neq \emptyset$, then $\mathbf{c}(\mathcal{B}') \subseteq \mathbf{c}(\mathcal{B})$.

Finally, suppose for the sake of contradiction that $\mathbf{c}(\mathcal{B}) \cap \mathbf{c}(\mathcal{B}') = \emptyset$. Let $Y \in \mathbf{c}(\mathcal{B})$ and $Z \in \mathbf{c}(\mathcal{B}')$, which are well-defined since the choice rule is nonempty valued. Notice that $Z \in \mathcal{B} \setminus \mathbf{c}(\mathcal{B})$ and $Y \in \mathcal{B}' \setminus \mathbf{c}(\mathcal{B}')$, so have $Y \ R^s_{\mathbf{c}} \ Z$ and $Z \ R^s_{\mathbf{c}} \ Y$, which contradicts WARP. □

Lemma 2. *For a single-valued classical choice model* $(\mathcal{X}, \mathscr{B}, \mathbf{c})$*, the following are equivalent:*

1. $\mathbf{c}$ *satisfies the Chernoff property.*
2. $\mathbf{c}$ *satisfies IIA.*
3. $\mathbf{c}$ *satisfies Arrow's axiom.*

Proof of Lemma 2. Let $\mathcal{B}, \mathcal{B}' \in \mathscr{B}$, where $\mathcal{B} \subseteq \mathcal{B}'$. First, suppose the Chernoff property holds, so $\mathbf{c}(\mathcal{B}') \cap \mathcal{B} \subseteq \mathbf{c}(\mathcal{B})$. Suppose $\mathbf{c}(\mathcal{B}') \subseteq \mathcal{B}$. Then, $\mathbf{c}(\mathcal{B}') \subseteq \mathbf{c}(\mathcal{B})$, and since $\mathbf{c}$ is single-valued (and nonempty valued), $\mathbf{c}(\mathcal{B}') = \mathbf{c}(\mathcal{B})$, so IIA is satisfied. Instead, suppose $\mathbf{c}(\mathcal{B}') \cap \mathcal{B} \neq \emptyset$. Since $\mathbf{c}$ is single-valued, $c(\mathcal{B}') \cap \mathcal{B} \neq \emptyset$ implies $\mathbf{c}(B') \subseteq \mathcal{B}$. Suppose IIA holds. Then $\mathbf{c}(\mathcal{B}) = \mathbf{c}(\mathcal{B}') = \mathbf{c}(\mathcal{B}') \cap \mathcal{B}$, so Arrow's axiom is satisfied. Next, Arrow's axiom is a strengthening of the Chernoff property, and so implies it (even without single-valuedness of $\mathbf{c}$). □

Proof of Theorem 1. [IRI implies WARP] Suppose that WARP is not satisfied, so that there exist $Z, Z' \in \mathcal{X}$ such that $Z \ R^s_{\mathbf{c}} \ Z'$ and $Z' \ R_{\mathbf{c}} \ Z$. Then, there exist $Y, Y' \subseteq X$ such that $Z, Z' \subseteq Y \cap Y'$, $C(Y) = Z$, and $C(Y') = Z'$. Now, $C(Y) = Z \subseteq Y \cap Y' \subseteq Y$ so, by IRI, $C(Y \cap Y') = C(Y) = Z$. Also, $C(Y') = Z' \subseteq Y \cap Y' \subseteq Y'$ so, by IRI, $C(Y \cap Y') = C(Y') = Z'$. But then I obtain $Z = Z'$, contradicting the hypothesis that $Z \ R^s_{\mathbf{c}} \ Z'$. Thus, WARP must hold if IRI is satisfied.

[WARP implies $R_C^B = R_{\mathbf{c}}$] It is clear that $R_C^B \subseteq R_{\mathbf{c}}$ even without WARP. Now, suppose $Z \; R_{\mathbf{c}} \; Z'$ for some $Z, Z' \in \mathcal{X}$, i.e., there exists $\mathcal{B} \in \mathscr{B}$ such that $Z, Z' \in \mathcal{B}$, $Z \in \mathbf{c}(\mathcal{B})$. Since $\mathscr{B}$ is a combinatorial domain, there exists $Y \subseteq X$ such that $\mathcal{B} = 2^Y$, so $Z \cup Z' \subseteq Y$. Since $\mathbf{c}$ is single-valued, for every $Z'' \in \mathcal{B}$ where $Z'' \neq Z$, $Z \; R_{\mathbf{c}}^s \; Z''$. In particular, for every $Z'' \in \mathcal{B}'$, where $\mathcal{B}' = 2^{Z \cup Z'} \subseteq \mathcal{B}$, it is the case that $Z \; R_{\mathbf{c}}^s \; Z''$. Then, by WARP, $\neg(Z'' \; R_{\mathbf{c}} \; Z)$. Thus, since $\mathbf{c}$ is not empty-valued, $\mathbf{c}(\mathcal{B}') = \{Z\}$, i.e. $C(Z \cup Z') = Z$, implying $Z \; R_C^B \; Z'$.

[$R_C^B = R_{\mathbf{c}}$ implies R_C^B-rationalizability] Let $\mathcal{B} \in \mathscr{B}$ and $Z \in \mathbf{c}(\mathcal{B})$. Then, for all $Z' \in \mathcal{B}$, $Z \; R_{\mathbf{c}} \; Z'$. Thus, by definition, $Z \in \mathbf{c}^{R_{\mathbf{c}}}(\mathcal{B})$. Hence, $\mathbf{c}(\mathcal{B}) \subseteq \mathbf{c}^{R_{\mathbf{c}}}(\mathcal{B}) = \mathbf{c}^{R_C^B}(\mathcal{B})$, where the latter equality follows by assumption. Next, let $Y \subseteq X$ and $\mathcal{B}' = 2^Y$. If $Z, Z' \in \mathbf{c}^{R_C^B}(\mathcal{B}')$, then by definition $Z \; R_C^B \; Z'$ and $Z' \; R_C^B \; Z$. The definition of the Blair relation implies that $Z = C(Z \cup Z')$ and $Z' = C(Z' \cup Z)$, so $Z = Z'$. Thus, $\mathbf{c}(\mathcal{B}) \supseteq \mathbf{c}^{R_C^B}(\mathcal{B})$. Thus, R_C^B rationalizes $\mathbf{c}$.

[R_C^B-rationalizability implies complete, reflexive, antisymmetric rationalizability] Let $\trianglerighteq$ be an arbitrary linear order on $\mathcal{X}$. Define a binary relation R on $\mathcal{X}$ by the following conditions: *a*) for each $Z \subseteq X$, $Z \; R \; Z$, *b*) for all $Z, Z' \subseteq X$ with $Z \neq Z'$, $Z \; R_C^B \; Z'$ implies $Z \; R \; Z'$ and $\neg(Z' \; R \; Z)$, and *c*) for all $Z, Z' \subseteq X$ with $Z \neq Z'$, if $\neg(Z \; R_C^B \; Z'$ or $Z' \; R_C^B \; Z)$, then $Z \; R \; Z'$ if and only if $Z \trianglerighteq Z'$. For Condition *b*) to be well-defined, R_C^B should be antisymmetric. This is ensured by C being single-valued. R is reflexive by Condition *a*), and Condition *b*) and Condition *c*) together ensure R is complete. Finally, R is antisymmetric because both R_C^B and $\trianglerighteq$ are antisymmetric.

Let $Y \subseteq X$ and $\mathcal{B} = 2^Y$. First, define $Z = C(Y)$. Since R_C^B rationalizes C, for all $Z' \subseteq Y$, $Z \; R_C^B \; Z'$, and so, by Condition *b*) of the definition of R, $Z \; R \; Z'$. Then, $Z \in \mathbf{c}^R(\mathcal{B})$, so $\mathbf{c}(\mathcal{B}) \subseteq \mathbf{c}^R(\mathcal{B})$.

Next, let $Z, Z' \in \mathbf{c}^R(\mathcal{B})$. Then, $Z \; R \; Z'$ and $Z' \; R \; Z$, so $Z = Z'$, by antisymmetry of R. So $|\mathbf{c}^R(\mathcal{B})| \leq 1$. Since $\mathbf{c}(\mathcal{B})$ is nonempty and a subset of $\mathbf{c}^R(\mathcal{B})$, we have $\mathbf{c}^R = \mathbf{c}$.

[complete, reflexive rationalizability implies rationalizability] This is immediate.

[rationalizability implies IRI] Let binary relation R rationalize $\mathbf{c}$. Let $Y \subseteq Y' \subseteq X$, $\mathcal{B} = 2^Y$, and $\mathcal{B}' = 2^{Y'}$. Let $Z' \in \mathbf{c}(\mathcal{B}')$, i.e. $Z' = C(Y')$. Then, rationalization by R implies $Z' \; R \; Z$ for all $Z \in \mathcal{B}'$. Thus, if $Z' \subseteq Y$, then $Z' \; R \; Z$ for all $Z \in \mathcal{B}$, and so $Z' \in \mathbf{c}(\mathcal{B})$. Since $\mathbf{c}$ is single-valued, $\mathbf{c}(\mathcal{B}) = \{Z'\} = \mathbf{c}(\mathcal{B}')$, i.e. $C(Y) = C(Y')$.

[WARP implies IIA] Follows from Lemma 1.

[IIA implies WARP] Let $Z, Z' \in \mathcal{X}$. Let $\mathcal{B} = 2^{Z \cup Z'} \in \mathscr{B}$. Since $\mathscr{B}$ is a combinatorial domain, for every $\tilde{\mathcal{B}} \in \mathscr{B}$, if $Z, Z' \in \tilde{\mathcal{B}}$, then $\mathcal{B} \subseteq \tilde{\mathcal{B}}$. Suppose $Z \; R_{\mathbf{c}}^s \; Z'$. Clearly, $Z \neq Z'$. Then there exists $\mathcal{B}' \in \mathscr{B}$ such that $Z, Z' \in \mathcal{B}'$, $Z \in \mathbf{c}(\mathcal{B}')$, and $Z' \notin \mathbf{c}(\mathcal{B}')$. Since $\mathbf{c}$ is single-valued, $\mathbf{c}(\mathcal{B}') = \{Z\}$. Then $\mathbf{c}(\mathcal{B}') \subseteq \mathcal{B} \subseteq \mathcal{B}'$, so by IIA, $\mathbf{c}(\mathcal{B}) = \mathbf{c}(\mathcal{B}') = \{Z\}$. Now suppose, for the sake of contradiction, that $Z' \; R_{\mathbf{c}} \; Z$, i.e., there exists $\mathcal{B}'' \in \mathscr{B}$ such that $Z, Z' \in \mathcal{B}''$ and $Z' \in \mathbf{c}(\mathcal{B}'')$. Since $\mathbf{c}$ is single-valued, $\mathbf{c}(\mathcal{B}'') = \{Z'\} \subseteq \mathcal{B} \subseteq \mathcal{B}''$. But since $Z \neq Z'$, $\mathbf{c}(\mathcal{B}'') \neq \mathbf{c}(\mathcal{B})$, contradicting IIA.

[IIA implies the Chernoff property implies Arrow's axiom implies IIA] Follows from Lemma 2.

[IRI implies path independence] Let $\mathcal{B}, \mathcal{B}' \in \mathscr{B}$, and let $Y, Y' \subseteq X$ be such that $\mathcal{B} = 2^Y$ and $\mathcal{B}' = 2^{Y'}$. Then, $\mathcal{B} \cup \mathcal{B}' \in \mathscr{B}$ if and only $2^Y \cup 2^{Y'} = 2^Z$ for some $Z \subseteq X$. This could only be if $Y \subseteq Y'$ or $Y' \subseteq Y$. So, the path independence condition applies only when considering opportunity sets ordered by inclusion. Moreover, it is also necessary that $\mathbf{c}(\mathcal{B}) \cup \mathcal{B}' \in \mathscr{B}$. Then, given that C is single-valued (and so $\mathbf{c}(\mathcal{B}) = \{C(Y)\}$), the path independence condition applies only when $C(Y) \in 2^{Y'} = \mathcal{B}'$, so that $\mathbf{c}(\mathcal{B}) \cup \mathcal{B}' = \mathcal{B}' \in \mathscr{B}$.

Therefore, path independence for the representation is equivalent to the following condition on the combinatorial choice function: for $Y, Y' \subseteq X$ with $Y \subseteq Y'$, $C(Y \cup Y') = C(C(Y) \cup Y')$ whenever $C(Y) \subseteq Y'$, and $C(Y \cup Y') = C(Y \cup C(Y'))$ whenever $C(Y') \subseteq Y$. The first part of this condition is trivially satisfied by every combinatorial choice function. Now, suppose that C satisfies IRI. Then, since $Y \subseteq Y'$, if $C(Y') \subseteq Y$ then $C(Y) = C(Y')$. But then $C(Y \cup Y') = C(Y') = C(Y) = C(Y \cup C(Y'))$, so obtain the second part of the condition.

[Path independence implies IRI] Let $Y \subseteq Y' \subseteq X$ and suppose $C(Y') \subseteq Y$. Then $C(Y') = C(Y \cup Y') = C(Y \cup C(Y'))$, where the latter equality follows from path independence given the assumption that $C(Y') \subseteq Y$. But the same assumption yields the conclusion that $C(Y \cup C(Y')) = C(Y)$ and so path independence implies IRI. □

The proof of Proposition 1 relies upon the following lemma. A *tie-breaker* τ for a classical choice model $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ is a complete, transitive, and asymmetric binary relation on $\mathcal{X}$. Let $\mathbf{c}^\tau$ be the tie-broken selection from $\mathbf{c}$ using tie-breaker τ, defined as follows: for every $\mathcal{B} \in \mathscr{B}$, $Z \in \mathbf{c}^\tau(\mathcal{B})$ if and only if $Z \in \mathbf{c}(\mathcal{B})$ and for every $Z' \in \mathbf{c}(\mathcal{B})$, $Z' \neq Z$ implies $Z \ \tau \ Z'$.

Lemma 3. *Let $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ be a classical choice model. Then $\mathbf{c}$ satisfies WARP if and only if for every tie-breaker τ the tie-broken selection $\mathbf{c}^\tau$ satisfies WARP.*[13]

Proof of Lemma 3. [WARP for $\mathbf{c}$ implies WARP for all $\mathbf{c}^\tau$] Suppose $\mathbf{c}$ satisfies WARP. Let τ be a tie-breaker and let $\mathbf{c}^\tau$ be the tie-broken selection from $\mathbf{c}$. Let $\mathcal{B} \in \mathscr{B}$, $Z = \mathbf{c}^\tau(\mathcal{B})$, $Z' \in \mathcal{B}$, and $Z' \neq Z$. For $\mathbf{c}^\tau$ to satisfy WARP, it is necessary that $\mathbf{c}^\tau(\mathcal{B}') \neq Z'$ for every $\mathcal{B}' \in \mathscr{B}$ such that $Z, Z' \in \mathcal{B}'$.

The first case is where $Z' \in \mathbf{c}(\mathcal{B})$. Then, by definition of $\mathbf{c}^\tau$, it must be that $Z \ \tau \ Z'$. Now, consider any $\mathcal{B}' \in \mathscr{B}$ such that $Z, Z' \in \mathcal{B}'$. Since $\mathbf{c}$ satisfies WARP, $Z \in \mathbf{c}(\mathcal{B}')$ if and only if $Z' \in \mathbf{c}(\mathcal{B}')$. Then, since $Z \ \tau \ Z'$, $\mathbf{c}^\tau(\mathcal{B}') \neq Z'$.

[13] Ehlers and Sprumont (2008) study the implications of the weakened weak axiom of revealed preference (WWARP), defined as follows: for every $Z, Z' \in \mathcal{X}$, $Z \ R^s_{\mathbf{c}} \ Z'$ implies $\neg(Z' \ R^s_{\mathbf{c}} \ Z)$. It is clear that WWARP is equivalent to WARP when the choice rule is single-valued, and so any selection from a choice rule satisfies WARP if and only if it satisfies WWARP. Thus, the analog of Lemma 3 for WWARP is not true.

The second case is where $Z' \notin \mathbf{c}(\mathcal{B})$, so that $Z \; R^s_{\mathbf{c}} \; Z'$. Consider any $\mathcal{B}' \in \mathscr{B}$ such that $Z, Z' \in \mathcal{B}'$. Since $\mathbf{c}$ satisfies WARP, $\neg(Z' \; R_{\mathbf{c}} \; Z)$, and so $Z' \notin \mathbf{c}(\mathcal{B}')$, which immediately implies $\mathbf{c}^\tau(\mathcal{B}') \neq Z'$.

[WARP for all $\mathbf{c}^\tau$ implies WARP for $\mathbf{c}$] Suppose for every tie-breaker τ the tie-broken selection $\mathbf{c}^\tau$ from $\mathbf{c}$ satisfies WARP. Let $\mathcal{B} \in \mathscr{B}$, $Z \in \mathbf{c}(\mathcal{B})$, and $Z' \in \mathcal{B} \setminus \mathbf{c}(\mathcal{B})$, so that $Z \; R^s_{\mathbf{c}} \; Z'$. For $\mathbf{c}$ to satisfy WARP, it is necessary that $Z' \notin \mathbf{c}(\mathcal{B}')$ for every $\mathcal{B}' \in \mathscr{B}$ such that $Z, Z' \in \mathcal{B}'$, so that $\neg(Z' \; R_{\mathbf{c}} \; Z)$.

Consider $\mathcal{B}' \in \mathscr{B}$ such that $Z, Z' \in \mathcal{B}'$. Let τ be the tie-breaker where $Z' \tau Z \tau Z''$ for every $Z'' \in \mathcal{X} \setminus \{Z, Z'\}$. Then, $\mathbf{c}^\tau(\mathcal{B}) = Z$. Since $\mathbf{c}^\tau$ satisfies WARP, $\mathbf{c}^\tau(\mathcal{B}') \neq Z'$. But since Z' is the highest ranked bundle under τ, this implies that $Z' \notin \mathbf{c}(\mathcal{B}')$. □

Proof of Proposition 1. Since $(\mathcal{X}, \mathscr{B}, \mathbf{c})$ is the faithful representation of (X, C), by definition $\mathcal{X} = 2^X$, and so a tie-breaker for (X, C) is also a tie-breaker for $(\mathcal{X}, \mathscr{B}, \mathbf{c})$, and vice-versa. Then, given a tie-breaker τ for (X, C), it is straightforward that $(\mathcal{X}, \mathscr{B}, \mathbf{c}^\tau)$ is the faithful representation of (X, C^τ). With this observation, the proof obtains from Lemma 3 and Theorem 1. □

Proof of Proposition 2. [WARP implies R^B_C-transitive-rationalizability] Given WARP, by Theorem 1 C satisfies IRI and R^B_C rationalizes $(\mathscr{B}, \mathbf{c})$. Alkan (2002) shows that R^B_C is transitive when C satisfies IRI and substitutability.[14] Then, R^B_C transitively rationalizes $(\mathscr{B}, \mathbf{c})$.

[R^B_C-transitive-rationalizability implies linear order rationalizability] Since C is single-valued, R^B_C is antisymmetric. By the Szpilrajn extension theorem, there exists a linear order R on 2^X that extends R^B_C. By the following claim, $\mathbf{c}^R = \mathbf{c}^{R^B_C}$, and so by the assumption of rationalizability by R^B_C, $\mathbf{c}^R = \mathbf{c}$.

Claim: Let $\tilde{R}$ be an antisymmetric relation on 2^X and $\tilde{R}'$ be an extension of $\tilde{R}$. Suppose $\mathbf{c}^{\tilde{R}}$ is nonempty-valued. Then, $\mathbf{c}^{\tilde{R}}$ is single-valued and $\mathbf{c}^{\tilde{R}'} = \mathbf{c}^{\tilde{R}}$.

Proof of Claim: Let $\mathcal{B} \in \mathscr{B}$. First, nonempty-valued $\mathbf{c}^{\tilde{R}}$ and antisymmetry of $\tilde{R}$ implies that $\mathbf{c}^{\tilde{R}}$ is single-valued, since $Z, Z' \in \mathbf{c}^{\tilde{R}}(\mathcal{B})$ implies both $Z \; \tilde{R} \; Z'$ and $Z' \; \tilde{R} \; Z$, and so $Z = Z'$.

Let $Z \in \mathbf{c}^{\tilde{R}}$ and let $Z' \in \mathcal{B}$.

By definition of $\mathbf{c}^{\tilde{R}}$, $Z \; \tilde{R} \; Z'$, so by definition of an extension, $Z \; \tilde{R}' \; Z'$. Since Z' is arbitrary, $Z \in \mathbf{c}^{\tilde{R}'}(\mathcal{B})$, so $\mathbf{c}^{\tilde{R}}(\mathcal{B}) \subseteq \mathbf{c}^{\tilde{R}'}(\mathcal{B})$. Now suppose $Z' \neq Z$, and note that Z is well-defined since $\mathbf{c}^{\tilde{R}}$ is nonempty-valued. From above $Z \; \tilde{R} \; Z'$, so antisymmetry of $\tilde{R}$ implies $\neg(Z' \; \tilde{R} \; Z)$. But then the requirement for an extension implies that $\neg(Z' \; \tilde{R}' \; Z)$, and so $Z' \notin \mathbf{c}^{\tilde{R}}(\mathcal{B})$.

Therefore, since $\mathcal{B}$ is arbitrary, $\mathbf{c}^{\tilde{R}'} = \mathbf{c}^{\tilde{R}}$. ■

[linear order rationalizability implies SARP] See Richter (1966, 1971).

[SARP implies WARP] Immediate.[15] □

[14] In fact, Alkan (2002) shows that R^B_C is a join-semilattice on the range of C. Koshevoy (1999) and Johnson and Dean (2001) have analogous results for classical choice rules satisfying IIA and the Chernoff property.

[15] Aygün and Sönmez (2013) show that IRI implies SARP given substitutability.

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