When is manipulation all about the ones and twos?*

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Abstract

A rule is pairwise strategy-proof if groups of size one and two never have an incentive to manipulate. When agents have strict preferences over their own outcomes, I show that pairwise strategy-proofness even eliminates incentives for any group of agents to manipulate, therefore implying group strategy-proofness. It is also equivalent to Maskin monotonicity. I obtain the equivalence results assuming preference domains satisfy a richness condition. Decomposing richness into two parts, I explore what brings about the equivalence. The results apply to school choice and matching with contracts, indivisible object allocation, and economies with private or public goods with single-peaked preferences.

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1 Introduction

As a practical matter in the design of real-world allocation rules, *strategy-proofness*, the removal of incentives for individuals to engage in misrepresentation, is an important desideratum. For one thing, a strategy-proof rule is robust to the mechanism designer's assumptions about the informational environment of agents in settings with private values.[1] For another, policy makers can both assess the performance of a strategy-proof rule with respect to its stated goals, and use the revealed information towards addressing other policy concerns.[2]

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[1] Wilson (1987) makes the case for dispensing with strong assumptions about the informational environment of agents.

[2] Some leading studies that make use of data from strategy-proof rules include Abdulkadiroğlu et al. (2014) and Abdulkadiroğlu et al. (2017).

A strategy-proof rule also has the desirable feature of "leveling the playing field" between sincere agents, who in general report truthfully, and sophisticated agents, who attempt to take advantage of any manipulation opportunities, possibly at the expense of sincere agents (Pathak and Sönmez, 2008). This property of strategic fairness played a key role in the adoption of the student-optimal stable rule by the Boston Public Schools system, to the chagrin of some sophisticated parents who were adept at manipulating the prior rule.[3,4]

Nevertheless, while the student-optimal stable rule, and strategy-proof rules in general, are immune to manipulations by individuals, they may still be manipulated by a group of agents in concert. For instance, two sophisticated parents in Boston might be able to coordinate a joint manipulation of the student-optimal stable rule to their benefit. Perhaps even larger groups of parents may collude to manipulate the system in their favor. A designer concerned with eliminating the possibility of all such manipulations should adopt a rule satisfying the stringent incentive requirement of *group strategy-proofness*.[5] However, requiring immunity to arbitrary group manipulations is overkill if coordination by large groups is difficult, which is likely the case in many practical applications such as school choice. The natural first step to addressing concerns about joint misrepresentations to a strategy-proof rule would be to require immunity to profitable manipulations by pairs of agents, that is, require *pairwise strategy-proofness*.[6] What is the scope for designing such rules?

In this paper, I study this question in a framework that accommodates public as well as private goods, where agents have strict preferences over their own outcomes. My main contribution (Theorem 1) is to show that a rule designed to eliminate incentives for individuals and pairs to manipulate succeeds in eliminating such incentives for arbitrary groups as well, if the possible preferences of agents is sufficiently varied. That is, a pairwise strategy-proof rule is actually group strategy-proof. Moreover, these two incentive

[3]Abdulkadiroğlu and Sönmez (2003) introduce the mechanism design approach to school choice, call attention to the problems of the original Boston mechanism, and propose and study the student-optimal stable rule as a possible alternative. See Abdulkadiroğlu et al. (2005) for details on the adaptation of the student-optimal stable rule for school assignment for the Boston Public Schools system.

[4]One particular group of sophisticated parents were the West Zone Parents Group in Boston. According to Pathak and Sönmez (2008), this group would meet prior to the time of admissions to discuss strategies for preference submission to the old rule. Under the redesigned, strategy-proof rule, these sophisticated parents cannot gain from individual strategizing.

[5]A group of agents has an incentive to manipulate if there is a deviating profile of reports for these agents, keeping non-manipulators reports fixed, such that every agent in this group is weakly better off, and at least one is strictly better off.

[6]A pair of agents has an incentive to manipulate if there is a deviating profile of reports for these agents, keeping non-manipulators reports fixed, such that both agents are weakly better off, and at least one is strictly better off.

compatibility properties are equivalent to Maskin monotonicity (Maskin, 1999), a condition on a rule well-known to be necessary for every Nash equilibrium outcome of the preference-reporting game to coincide with the outcome from truthful reporting.

Theorem 1 has a number of applications. For the problem of school choice, the student-optimal stable rule is pairwise strategy-proof if and only if the schools' priorities satisfy a restrictive acyclicity condition due to Ergin (2002).[7] Unfortunately, acyclicity does not hold for priorities in most applications, and so there is little scope for eliminating manipulations by pairs of agents while preserving stability. For many well-known characterizations of indivisible goods allocation rules on the domain of strict preferences, Theorem 1 implies that the axiom of group strategy-proofness can be replaced with pairwise strategy-proofness. For instance, Pycia and Ünver (2017) characterize all group strategy-proof and Pareto efficient rules by what they call *trading cycles* rules, for invisible goods allocation with unit demand and supply. Theorem 1 implies that their characterization holds even if the requirement of group strategy-proofness is weakened to pairwise strategy-proofness.

I obtain this equivalence theorem assuming that each agent's domain of possible preferences is *rich*. This roughly requires that for every pair of possible preferences of an agent and every pair of outcomes for the agent, there is a preference in the domain that simultaneously reflects an improvement of the first outcome when judged by the first preference and an improvement of the second outcome when judged by the second preference.[8] Some examples of rich domains are: *a*) the set of all strict preferences over outcomes, *a*) the set of all strict single-peaked preferences over an exogenous order of outcomes, *a*) domains where, for every ordered pair of outcomes, there is preference relation that rank them first and second.

A secondary contribution is to get to root of the equivalence result in Theorem 1 by studying two novel preference domain conditions I call *two-point connectedness* and *one-point connectedness*. These connectedness requirements on the preference domain are jointly equivalent to richness. If any one of these two is not satisfied, it is possible to find a pairwise strategy-proof rule that is not group strategy-proof. Also, Maskin monotonicity generally does not imply pairwise strategy-proofness without both these requirements. However, with only two-point connectedness, I show that a pairwise strategy-proof rule is *weakly group strategy-proof* (Proposition 5), though the converse is not generally true.[9]

[7] Ergin (2002) proves that the student-optimal stable rule is group strategy-proof exactly when this condition is satisfied. That the student-optimal stable rule is the only candidate is due to Alcalde and Barberà (1994).

[8] The richness condition is the counterpart to the one in Dasgupta et al. (1979) on domains that also satisfy the requirement of strict preferences over own outcomes.

[9] Weak group strategy-proofness is weaker because it only requires immunity to group manipulations

In addition to economies with discrete private goods, the framework admits those that allow for public goods as well, including the general social choice (or voting) model. The assumption of strict preferences over own outcomes requires that an agent is indifferent between two allocations only if his outcome in each of these allocations is the same. So, if there are public goods, any change in the public good component must change the welfare of every agent. For the case of social choice with strict preferences, two-point connectedness, which is weaker than richness, is sufficient for the three-way equivalence of strategy-proofness, group strategy-proofness, and Maskin monotonicity (Corollary 2). A number of interesting subdomains of the single-peaked preference domain are two-point connected. Some of these have the property that no two admissible preferences have the same peak (Proposition 10), which makes them rather "small". Since the smaller the preference domain, the more powerful is such an equivalence result, Corollary 2 shows how far the result of Muller and Satterthwaite (1977), that strategy-proofness and Maskin monotonicity are equivalent when all preferences are admissible, extends.

Related Literature: Serizawa (2006) defines a weakening of pairwise strategy-proofness, *effective pairwise strategy-proofness*, which insists on robustness of a pairwise manipulation to a further deviation by one of the manipulators. He shows, *inter alia*, that effective pairwise strategy-proofness is equivalent to group strategy-proofness on the classical preference domain in economies with one public and one private divisible good.[10] I contribute to the study of this axiom by examining its implications for private good economies without transfers and with strict preferences over own outcomes. I establish that effective pairwise strategy-proofness is equivalent to strategy-proofness and non-bossiness without any richness requirement, and is equivalent to group strategy-proofness for rich domains. Serizawa's result and mine are independent, because his preference domain allows indifferences but places the restrictions on the upper-level sets.[11]

There are a number of studies that identify conditions under which strategy-proofness is equivalent to a group incentive-compatibility requirement. Mostly closely related is the result of Pápai (2000) that strategy-proofness and group strategy-proofness are equivalent for object allocation with unit demand, when the rule is non-bossy and the domain comprises all strict preference profiles. The main result in the present paper extends

where every member of the manipulating group is made strictly better off.

[10]The classical preference domain for this class of economies is the set of preferences that are representable by a continuous utility function that is strictly quasiconcave and strictly monotonic on the interior of $\mathbb{R}^2_+$.

[11]Serizawa (2006) also studies exchange economies and the allotment problem with single-peaked preferences. Effective pairwise strategy-proofness implies group strategyproofness when efficiency is assumed in the case of exchange economies and unanimity in the case of allotment problems.

this equivalence to a more general model, while weakening the assumption on the domain to that of richness, allowing for novel applications to models with single-peaked preferences.

Another well-studied group incentive compatibility requirement is that of weak group strategy-proofness. Barberà and Jackson (1995) show the equivalence of strategy-proofness and weak group strategy-proofness for non-bossy rules in exchange economies with the classical preference domain. More recently, Barberà et al. (2016) show such an equivalence for general private good economies. They allow for indifferences but use a stronger richness condition than the one I employ. Their equivalence result is obtained under two maintained assumptions on the rule that weaken non-bossiness and Maskin monotonicity. For the social choice setting, Le Breton and Zaporozhets (2009) and Barberà et al. (2010) furnish other domain conditions that guarantee equivalence between strategy-proofness and weak group strategy-proofness.

There also exists a literature on the connection between Maskin monotonicity and group strategy-proofness in settings with private goods. Most closely related is Takamiya (2007), who shows the equivalence of these two properties in a similar framework to mine. I improve upon his result by showing equivalence with effective pairwise strategy-proofness. Klaus and Bochet (2013) study the relation between strategy-proofness and both Maskin monotonicity and a weaker variant in a similar framework, but allowing for weak preferences. They introduce a domain condition to obtain equivalence results between these properties. This condition is neither stronger nor weaker than richness in this paper, when imposed on domains of strict preferences over own outcomes. Moreover, for certain single-peaked preference domains in the social choice setting, two-point connectedness is a strictly less restrictive requirement than the condition they introduce.

The rest of the paper is organized as follows. I define the model and properties of rules in Section 2. I present the main results for pairwise strategy-proofness in Section 3. I discuss applications of the results in Section 4, and conclude in Section 5. In Appendix A, I prove results concerning manipulations by groups up to size $m \in \mathbb{Z}_+$, which generalize the results in Section 3. In Appendix B, I prove results presented in the body of the paper.

2 Preliminaries

2.1 The Elements of the Model

Let N be a finite and nonempty set of **agents**. For each $i \in N$, let X_i be a nonempty set of possible **outcomes** for i. Let $\mathcal{A} \subseteq \times_{i \in N} X_i$ be a nonempty set of **allocations**. Let α_i

denote the outcome for $i \in N$ in allocation $\alpha \in \mathcal{A}$. An outcome could be a singleton or a larger set of objects, a social alternative, a coalition of agents that includes i, a contract in a match involving i, or generally a list of one or more public or private goods, divisible or indivisible.

Public goods can be modeled because the allocation set $\mathcal{A}$ need not be a Cartesian product of outcomes. For instance, consider an economy with one private good and one public good, where an allocation is a list $(z_1, \ldots, z_N, y)$, with $z_i \in Z_i$ the private good allotment of agent $i \in N$, and $y \in Y$ the level of the public good. Then model the outcome of agent i by (z_i, y), and the set of outcomes for i by $X_i = Z_i \times Y$. The set of allocations $\mathcal{A}$ is now a subset of $\times_{i \in N}(Z_i \times Y)$, where for every $\alpha \in \mathcal{A}$ and $i, j \in N$, the Y-component of α_i and α_j agree.

Let $i \in N$. Define $\tilde{\mathcal{P}}_i$ to be the set of all linear orderings on the set X_i.[12] For any $x, x' \in X_i$ and $R_i \in \tilde{\mathcal{P}}_i$, $x \; R_i \; x'$ represents that i finds x to be at least as good as x'. Let P_i be the asymmetric component that represents the *strict* preference of i, defined as follows: $x \; P_i \; x'$ if and only if $x \; R_i \; x'$ and $x \neq x'$. Note that P_i uniquely identifies R_i, so I will interchange references to P_i and R_i.

I assume throughout that preferences of agents over allocations are determined by their preferences over their component in the allocation, so that for any $\alpha, \alpha' \in \mathcal{A}$, agent $i \in N$ prefers α to α' if and only if he prefers α_i to α'_i. Abusing notation, let R_i also denote agent i's preferences over $\mathcal{A}$.If he is indifferent between allocations α and α', then it must be that $\alpha_i = \alpha'_i$.[13]

For every $i \in N$, call $\mathcal{P}_i \subseteq \tilde{\mathcal{P}}_i$ a **preference domain for** i. Let $\tilde{\mathcal{P}} = \times_{i \in N} \tilde{\mathcal{P}}_i$. Each $R \in \tilde{\mathcal{P}}$ is a *preference profile*. Call $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ a **domain of preference profiles**. A domain $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ is **Cartesian** if there exists for every $i \in N$ a preference domain $\mathcal{P}_i$ such that $\mathcal{P} = \times_{i \in N} \mathcal{P}_i$. Unless otherwise specified, all preference domains I consider will be Cartesian.

A **rule** φ is a function from $\mathcal{P}$ to $\mathcal{A}$. For every $R \in \mathcal{P}$ and $i \in N$, let $\varphi_i(R) = \alpha_i$, where $\alpha = \varphi(R)$.

2.2 Properties of Rules

Given a rule φ, a set $M \subseteq N$ *manipulates* φ *at* $R \in \mathcal{P}$ *via* $R' \in \mathcal{P}$, if a) for every $i \in N \setminus M$, $R'_i = R_i$, a) for every $l \in M$, $\varphi_l(R') \; R_l \; \varphi_l(R)$, and a) there exists $l \in M$, $\varphi_l(R') \; P_l \; \varphi_l(R)$. In this case, (M, R, R') is a **manipulation** of φ. Condition a) ensures that the move from profile R

[12] A linear ordering is a complete, reflexive, transitive, and antisymmetric binary relation.

[13] To be more precise, in this case R_i is a complete, reflexive, and transitive binary relation, though not necessarily antisymmetric.

to R' only involves a change in the preferences of agents in M.[14] Additionally, not every manipulator needs to be made strictly better off. If $\varphi_l(R')\ P_l\ \varphi_l(R)$ for every $l \in M$, then (M, R, R') is a **strong manipulation**. A manipulation (M, R, R') of a rule φ is **robust** if for every $i \in M$ and for every $\hat{R}_i \in \mathcal{P}_i$, $\varphi_i(R')\ R_i\ \varphi_i(\hat{R}_i, R'_{-i})$. A robust manipulation is immune to a further deviation by individual manipulators, thereby imposing stronger demands on the group seeking to profitably misrepresent preferences.

A rule φ is **strategy-proof** if there does not exist an agent $i \in N$ and $R, R' \in \mathcal{P}$ such that $(\{i\}, R, R')$ is a manipulation. It is **group strategy-proof** if there does not exist a set $M \subseteq N$ and $R, R' \in \mathcal{P}$ such that (M, R, R') is a manipulation. It is **weakly group strategy-proof** if there does not exist a set $M \subseteq N$ and $R, R' \in \mathcal{P}$ such that (M, R, R') is a strong manipulation. It is **pairwise strategy-proof** if there does not exist a set $M \subseteq N$, $|M| \leq 2$, and $R, R' \in \mathcal{P}$ such that (M, R, R') is a manipulation. It is **effectively pairwise strategy-proof** (Serizawa, 2006) if there does not exist a robust manipulation (M, R, R') with $|M| \leq 2$.[15] This is a weaker requirement than pairwise strategy-proofness.

Let $i \in N$, $x \in X_i$, and $R_i, R'_i \in \tilde{\mathcal{P}}_i$. The preference relation R'_i is a **monotonic transformation of** R_i **at** x if for every $z \in X_i$, $x\ R_i\ z$ implies $x\ R'_i\ z$. Denote the set of all monotonic transformations of R_i at x by $MT_i(R_i, x)$. Clearly, for every $x \in X_i$ and $R_i \in \tilde{\mathcal{P}}_i$, $R_i \in MT_i(R_i, x)$. Let $R, R' \in \times_{i \in N} \tilde{\mathcal{P}}_i$. Preference profile R' is a *monotonic transformation of R at allocation α* if $R'_i \in MT_i(R_i, \alpha_i)$ for every $i \in N$. A rule φ is **Maskin monotonic**, i.e. invariant to monotonic transformations, if for every $R \in \mathcal{P}$ and every $R' \in \mathcal{P}$ that is a monotonic transformation of R at $\varphi(R)$, $\varphi(R') = \varphi(R)$.

A rule φ is **non-bossy** if for every profile $R \in \mathcal{P}$, for every agent $i \in N$, and every preference relation $\tilde{R}_i \in \mathcal{P}_i$, $\varphi_i(R) = \varphi_i(\tilde{R}_i, R_{-i})$ implies $\varphi(R) = \varphi(\tilde{R}_i, R_{-i})$. It is **group non-bossy** if for every profile $R \in \mathcal{P}$, for every set of agents $M \subseteq N$, and every preference profile $\tilde{R}_M \in \mathcal{P}_M$, if $\varphi_i(R) = \varphi_i(\tilde{R}_M, R_{-M})$ for every $i \in M$, then $\varphi(R) = \varphi(\tilde{R}_M, R_{-M})$.[16]

Each property of interest can be classified as a strategic or an invariance property. The strategic ones are, in increasing strength, strategy-proofness, effective pairwise strategy-proofness, pairwise strategy-proofness, and group strategy-proofness. The invariance ones are non-bossiness, group non-bossiness, and Maskin monotonicity. Group non-bossiness is stronger than non-bossiness, but in general neither property is implied by

[14] I use this definition of a manipulation to reduce the notational burden, and it should be understood that condition *a*) applies whenever two profiles that constitute a manipulation are referenced.

[15] Schummer (2000) considers *bribe-proofness*, the immunity to a particular pairwise manipulation where an agent pays another to misreport preferences to the benefit of both. While in the spirit of pairwise strategy-proofness, it is defined for environments with a transferable good with quasilinear preferences, which I do not consider.

[16] This generalization of non-bossiness is also studied in Afacan (2012), who shows that group strategy-proofness implies group non-bossiness in the classic object allocation model.

or implies Maskin monotonicity.

3 When is manipulation all about the ones and twos?

I introduce two novel domain conditions that feature in the subsequent results relating various incentive and invariance properties of a rule. I then present the main equivalence theorem (Theorem 1), followed by a series of results relating properties of a rule that demonstrate the theorem. I discuss applications in the next section. I extend the analysis to manipulations involving groups of up to size $m \in \mathbb{Z}_+$ in Appendix A. Proofs of all the results are in Appendix B.

3.1 Domain conditions

The first domain condition, **two-point connectedness**, is so-called because it relates two upper-level sets at two distinct points in the outcome space. A preference domain $\mathcal{P}_i \subseteq \tilde{\mathcal{P}}_i$ for $i \in N$ is two-point connected if for every $\alpha, \beta \in \mathcal{A}$ and for every $R_i, R_i' \in \mathcal{P}_i$ such that $\alpha_i \; P_i \; \beta_i$, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $\hat{R}_i \in MT_i(R_i', \alpha_i) \cap MT_i(R_i, \beta_i)$. A domain of preference profiles $\mathcal{P}$ is *two-point connected* if $\mathcal{P}_i$ is two-point connected for every $i \in N$.

The set of all strict preferences and the set of all strict single-peaked preferences for a given ordering of discrete outcomes both satisfy two-point connectedness. Unlike the set of all strict single-peaked preferences, the set of all strict single-dipped preferences do not satisfy two-point connectedness, except when Remark 1 below applies. Another example of a two-point connected domain for agent i is one that allows any pair of distinct outcomes in X_i to occupy the first and second rank under some preference relation, a strictly stronger requirement. Formally, $\mathcal{P}_i$ has *unrestricted top pairs* if for every distinct pair $x, y \in X_i$, there exists $R_i \in \mathcal{P}_i$ such that $x \; P_i \; y \; P_i \; z$ for every $z \in X_i \setminus \{x, y\}$.[17]

Remark 1. *Two-point connectedness is satisfied when the number of outcomes for each agent does not exceed three: if $|X_i| \leq 3$ for agent i, then every $\mathcal{P}_i \subseteq \tilde{\mathcal{P}}_i$ is 2-point connected.*

The second domain condition is called **one-point connectedness**, because it relates two upper-level sets at one point in the outcome space. A preference domain $\mathcal{P}_i \subseteq \tilde{\mathcal{P}}_i$ for $i \in N$ is one-point connected if for every $\alpha \in \mathcal{A}$ and for every $R_i, R_i' \in \mathcal{P}_i$, there exists $\hat{R} \in \mathcal{P}_i$ such that $\hat{R} \in MT_i(R_i, \alpha_i) \cap MT_i(R_i', \alpha_i)$. A domain of preference profiles $\mathcal{P}$ is *one-point connected* if $\mathcal{P}_i$ is one-point connected for every $i \in N$.

[17] In fact, such a domain also satisfies one-point connectedness, and hence richness, defined below.

Many domains satisfy one-point connectedness. A simpler yet stronger requirement than one-point connectedness is the following: $\mathcal{P}_i$ has *unrestricted tops* if for every $x \in X_i$, there exists $R_i \in \mathcal{P}_i$ such that $x \; P_i \; z$ for all $z \in X_i \setminus \{x\}$. For example, any subdomain of the set of all single-peaked preferences on an ordered interval that contains, for every outcome, some preference with that outcome as peak satisfies one-point connectedness since it has unrestricted tops. However, the set of strict single-dipped preferences is not one-point connected except in the uninteresting case when there is at most two possible outcomes.

Dasgupta et al. (1979) introduce a *richness* condition for a model with weak preferences. I define a version that is equivalent to theirs when imposed on any Cartesian subdomain of $\tilde{\mathcal{P}}$. A preference domain $\mathcal{P}_i \subseteq \tilde{\mathcal{P}}_i$ for $i \in N$ is **rich** if for every $\alpha, \beta \in \mathcal{A}$ and for every $R_i, R_i' \in \mathcal{P}_i$ such that $\alpha_i \; R_i \; \beta_i$, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $\hat{R}_i \in MT_i(R_i', \alpha_i) \cap MT_i(R_i, \beta_i)$. A domain of preference profiles $\mathcal{P}$ is *rich* if $\mathcal{P}_i$ is rich for every $i \in N$.

Remark 2. *The gap between richness and two-point connectedness is exactly given by one-point connectedness, i.e., a preference domain $\mathcal{P}_i \subseteq \tilde{\mathcal{P}}_i$ for $i \in N$ is rich if and only if it is both one- and two-point connected.*

A number of interesting domains of preferences satisfy this richness condition. For example, the set of all possible strict preferences $\tilde{\mathcal{P}}_i$ is a rich domain. Another important example is the domain of all single-peaked strict preferences on a discrete ordered set.

Remark 3. *A Cartesian subdomain of a rich (alternatively, two-point connected) domain may not be rich (two-point connected).*

Richness is a stronger domain condition that two-point connectedness. Some strict subsets of the set of strict single-peaked preferences satisfy two-point connectedness, but not richness.[18] Section 4 contains an economic application where two-point connectedness, but not richness, is satisfied.

Other domain conditions Fleurbaey and Maniquet (1997) define a condition, monotonic closedness, on domains drawn from the set of all weak orders. On subdomains of $\tilde{\mathcal{P}}$, monotonic closedness is equivalent to two-point connectedness. Condition R1 in Klaus and Bochet (2013) applied to subdomains from $\tilde{\mathcal{P}}$ implies two-point connectedness but is a more demanding requirement, because it requires that every outcome be top-ranked under some preference.[19] It is not equivalent even when the set of allocations is finite

[18] For instance, in Example 3, $\mathcal{P}_i$ is a domain of strict preferences that are single-peaked over the ordered set $x < y < z$ and that satisfies two-point connectedness but not richness.

[19] Since X_i can be infinite, a preference relation R_i need not have a well-defined "best" outcome in X_i.

(see Example 4, for instance). Condition R1 also implies one-point connectedness unless there is an always-worst alternative, i.e. if $\mathcal{P}_i$ satisfies Condition R1 but not one-point connectedness, then there exists $\alpha \in \mathcal{A}$ such that for every $\beta \in \mathcal{A}$ and every $R_i \in \mathcal{P}_i$, $\beta \; R_i \; \alpha$. In fact, there are domains that are rich yet fail to satisfy Condition R1 (Proposition 10). Barberà et al. (2012) define a condition, intertwinedness, on domains drawn from the set of all weak orders. Every intertwined subdomain from $\tilde{\mathcal{P}}$ is two-point connected. However, there are two-point connected domains that are not intertwined.[20]

3.2 Results

The main equivalence result of the paper is the following theorem.

Theorem 1. *Let $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ be a Cartesian domain of preference profiles, and let φ be a rule on $\mathcal{P}$. If $\mathcal{P}$ is one-point and two-point connected (i.e. rich), then the following statements are equivalent:*

1. *φ is effectively pairwise strategy-proof.*
2. *φ is pairwise strategy-proof.*
3. *φ is group strategy-proof.*
4. *φ is strategy-proof and non-bossy.*
5. *φ is strategy-proof and group non-bossy.*
6. *φ is Maskin monotonic.*

I will obtain this equivalence result through a series of propositions about various incentive and invariance properties under increasingly strong domain requirements.

I begin with results for arbitrary Cartesian domains of preference profiles. First, effective pairwise strategy-proofness is equivalent to the combination of strategy-proofness and non-bossiness.

Proposition 1. *Let $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ be a Cartesian domain of preference profiles, and let φ be a rule on $\mathcal{P}$. Then the following statements are equivalent:*

1. *φ is effectively pairwise strategy-proof.*
2. *φ is strategy-proof and non-bossy.*

Non-bossiness is a commonly used axiom in the analysis of rules, often in conjunction with strategy-proofness. Its value, in part, stems from technical convenience. However,

[20] For intertwined domains, these authors show that a weak monotonicity condition together with another condition called reshuffling invariance is equivalent to strategy-proofness.

Thomson (2016) criticizes the use of non-bossiness as a strategic property, arguing that it "makes sense only when imposed on rules that are *strategy-proof*, in fact *pairwise strategy-proof*". For the class of models I study, this critique can be avoided, since Proposition 1 provides a conceptual foundation for its use: the axiom of effective pairwise strategy-proofness can always replace the combination of strategy-proofness and non-bossiness.

For object allocation problems with unit demand and the complete domain of strict preferences $\tilde{\mathcal{P}}$, a well-known equivalence is that of group strategy-proofness with the combination of strategy-proofness and non-bossiness (Pápai, 2000). However, this equivalence does not hold for arbitrary domains. In fact, effective pairwise strategy-proofness cannot even be replaced by pairwise strategy-proofness in the statement of Proposition 1, as is clear from Example 1.

Example 1. *Let* $N = \{i, j\}$, $X_i = \{w, x, y, z\}$, *and* $X_j = \{a, b, c\}$. *Let* $\mathcal{P}_i = \{R_i, R'_i\}$ *and* $\mathcal{P}_j = \{R_j, R'_j\}$ *satisfy:*

R_i	R'_i	R_j	R'_j
w	z	a	c
x	y	b	b
y	x	c	a
z	w		

Consider the rule φ *on* $\mathcal{P}$ *defined as follows:*

φ	R_i	R'_i
R'_j	w, c	x, b
R_j	y, b	z, a

This rule is effectively pairwise strategy-proof (and even weakly group strategy-proof). However, $(\{i, j\}, R, R')$ *is a non-robust manipulation, where* $R = (R_i, R_j)$ *and* $R' = (R'_i, R'_j)$. *Thus* φ *is not pairwise strategy-proof.*

Note that $\mathcal{P}_i$ *is not two-point connected — even though* $x \; P_i \; y$, *there does not exist* $\hat{R}_i \in \mathcal{P}_i$ *such that* $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R'_i, x)$.

The next result for arbitrary domains states that effective pairwise strategy-proofness implies Maskin monotonicity.

Proposition 2. *Let* $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ *be a Cartesian domain of preference profiles, and let* φ *be a rule on* $\mathcal{P}$. *If* φ *is effectively pairwise strategy-proof, then it is Maskin monotonic.*[21]

[21] Proposition 2 improves upon Theorem 1 in Takamiya (2007), which shows that group strategy-proofness implies Maskin monotonicity in a model isomorphic to this paper's. In fact, there is a slight gap in the proof in Takamiya (2007), because he only proves that group strategy-proofness implies *individual monotonicity* (Takamiya, 2001), rather than Maskin monotonicity.

In light of Propositions 1 and 2, a strategy-proof and non-bossy rule is Maskin monotonic for any domain. The converse of Proposition 2 is not true without both two- and one-point connectedness, as shown in Example 2.[22]

Example 2. *Let $N = \{i, j\}$, $X_i = \{x, y, z\}$, and $X_j = \{a, b, c\}$. Let $\mathcal{P}_i = \{R_i, R_i'\}$ and $\mathcal{P}_j = \{R_j, R_j'\}$ satisfy:*

R_i	R_i'	R_j	R_j'
x	z	a	c
y	y	b	b
z	x	c	a

Consider the rule φ on $\mathcal{P}$ defined as follows:

R_j'	y, b	z, b
R_j	y, b	y, b
φ	R_i	R_i'

The domain is two-point connected (but not one-point connected) and this rule is not effectively pairwise strategy-proof (and fails to be non-bossy) since $(\{i, j\}, (R_i', R_j), R')$ is a robust manipulation.

Since for every distinct $l, l' \in N$, every $R_{l'} \in \mathcal{P}_{l'}$, and every distinct pair $R_l, R_l' \in \mathcal{P}_l$, $R_l' \notin MT(R_l, \varphi_l(R_l, R_{l'}))$, the rule φ is Maskin monotonic.

Now, I obtain stronger results for domains that satisfy two-point connectedness. The first result with two-point connectedness is that ruling out robust pairwise manipulations actually rules out any pairwise manipulation.

Proposition 3. *Let $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ be a Cartesian domain of preference profiles, and let φ be a rule on $\mathcal{P}$. If $\mathcal{P}$ is two-point connected, then the following statements are equivalent:*

1. *φ is effectively pairwise strategy-proof.*
2. *φ is pairwise strategy-proof.*

However, even if two-point connectedness fails for some agent, if every other agent has no more than two outcomes, the equivalence in Proposition 3 obtains.

Proposition 4. *Suppose the following holds: if there exists $i \in N$ such that $|X_i| > 3$, then for every $j \in N \setminus \{i\}$, $|X_j| \leq 2$. Let $\mathcal{P} = \times_{i \in N} \mathcal{P}_i$ be a Cartesian domain of preference profiles. A rule φ on $\mathcal{P}$ is effectively pairwise strategy-proof if and only if it is pairwise strategy-proof.*

[22]Two-point connectedness is satisfied in the example, illustrating the need for one-point connectedness as well. Klaus and Bochet (2013) show the equivalence of Maskin monotonicity with strategy-proofness and non-bossiness, for domains satisfying their Conditions R1 and R2 (Theorem 3, part (c) in their paper). As discussed earlier, Condition R1 applied to a strict preference domain with a finite set of alternatives is neither weaker nor stronger than richness.

Proposition 4 is tight in the following sense: if there are at least four distinct outcomes for some agent and at least three distinct outcomes for another, then there exists a rule that is effectively pairwise strategy-proof but not pairwise strategy-proof. This can be seen in Example 1, where two-point connectedness does not hold.

Two-point connectedness alone is not sufficient for the equivalence of effectively pairwise and group strategy-proofness. This can be seen in Example 3, where $\mathcal{P}_i$ and $\mathcal{P}_j$ are two-point connected (but not one-point connected at outcomes y and b, respectively). It is possible to construct a similar example showing that one-point connectedness by itself is not sufficient either.

Example 3. *Let $N = \{i,j,k\}$, $X_i = \{x,y,z\}$, $X_j = \{a,b,c\}$, and $X_k = \{u,v\}$. Let $\mathcal{P}_i = \{R_i, R_i'\}$, $\mathcal{P}_j = \{R_j, R_j'\}$, and $\mathcal{P}_k = \{R_k\}$ such that:*

R_i	R_i'	R_j	R_j'	R_k
x	z	a	c	u
y	y	b	b	v
z	x	c	a	

Consider the rule φ on $\mathcal{P}$ defined as follows:

R_j'	x,c,v	y,b,u
R_j	y,b,v	z,a,v
$\varphi : R_k$	R_i	R_i'

This rule is pairwise strategy-proof. However, $(\{i,j,k\}, R, R')$ is a manipulation involving three agents, where $R = (R_i, R_j, R_k)$ and $R' = (R_i', R_j', R_k)$, and so is not group strategy-proof.

Note that $\mathcal{P}_i$ (and analogously $\mathcal{P}_j$) is two-point connected but not one-point connected — there does not exist $\hat{R}_i \in \mathcal{P}_i$ such that $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R_i', y)$.

Even though (effective) pairwise strategy-proofness does not imply group strategy-proofness when the domain is only two-point connected, it does imply weak group strategy-proofness.

Proposition 5. *Let $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ be a Cartesian domain of preference profiles, and let φ be a rule on $\mathcal{P}$. Suppose $\mathcal{P}$ is two-point connected. If φ is (effectively) pairwise strategy-proof, it is weakly group strategy-proof.*

The converse, however, does not hold true. The rule in the following example has a two-point connected domain and is weakly group strategy-proof but not effectively pairwise strategy-proof.

Example 4. *Modify the rule φ from Example 2 as follows:*

φ	R_i	R'_i
R'_j	x, c	y, b
R_j	z, b	z, a

This rule is weakly group strategy-proof, and $\mathcal{P}$ is two-point connected by Remark 1. However, $(\{i,j\}, R, (R'_i, R_j))$ is a robust manipulation, where $R = (R_i, R_j)$. Thus φ is not even effectively pairwise strategy-proof.

As noted earlier, Example 2 shows that two-point connectedness is not sufficient for Maskin monotonicity to imply (effective) pairwise strategy-proofness. However, it is sufficient for Maskin monotonicity to imply strategy-proofness.

Proposition 6. *Let $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ be a Cartesian domain of preference profiles, and let φ be a rule on $\mathcal{P}$. Suppose $\mathcal{P}$ is two-point connected. If φ is Maskin monotonic, then it is strategy-proof.*

Example 5 shows that Proposition 6 fails without two-point connectedness.

Example 5. *Modify the rule φ from Example 1 as follows:*

φ	R_i	R'_i
R'_j	y, b	x, b
R_j	y, b	y, b

The domain $\mathcal{P}_i$ is not two-point connected, and this rule φ is not strategy-proof since $(\{i\}, (R_i, R'_j), R')$ is an individual manipulation.

Since for every distinct $l, l' \in N$, every $R_{l'} \in \mathcal{P}_{l'}$, and every distinct pair $R_l, R'_l \in \mathcal{P}_l$, $R'_l \notin MT(R_l, \varphi_l(R_l, R_{l'}))$, the rule φ is Maskin monotonic.

The final result for two-point connected domains is a counterpart to Proposition 1. It states the equivalence between group strategy-proofness and the combination of strategy-proofness and group non-bossiness.

Proposition 7. *Let $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ be a Cartesian domain of preference profiles, and let φ be a rule on $\mathcal{P}$. If $\mathcal{P}$ is two-point connected, then the following statements are equivalent:*

1. *φ is group strategy-proof.*
2. *φ is strategy-proof and group non-bossy.*

As I show in Appendix A, non-bossiness and group non-bossiness are generally not equivalent properties of a rule. Example 1 makes clear that two-point connectedness cannot be easily dispensed with, since in a two agent economy non-bossiness immediately implies group non-bossiness.

Finally, I offer two results for one-point connected domains. The first one states that non-bossiness and group non-bossiness are equivalent properties of a strategy-proof rule on a one-point connected domain.

Proposition 8. *Let $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ be a Cartesian domain of preference profiles, and let φ be a rule on $\mathcal{P}$. If $\mathcal{P}$ is one-point connected and φ is strategy-proof, then non-bossiness and group non-bossiness are equivalent.*

The second result states that Maskin monotonicity implies group non-bossiness.

Proposition 9. *Let $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ be a Cartesian domain of preference profiles, and let φ be a rule on $\mathcal{P}$. If $\mathcal{P}$ is one-point connected and φ is Maskin monotonic, then it is group non-bossy.*

Examples 3 and 2 make clear that Propositions 8 and 9, respectively, fail without one-point connectedness, even given two-point connectedness.

To conclude this section, Theorem 1 is a consequence of Propositions 1 to 8. From an examination of the proofs it can be seen that, given a particular rule, the two domain conditions need only hold for those outcomes that are in the range of the rule, and so are not in a strict sense necessary. Nevertheless, as the examples above make clear, particularly Example 3, it is not possible to significantly weaken the domain requirements and preserve the equivalence theorem.

4 Applications

Social choice The classic social choice model has a set of social alternatives A, and agents have strict preferences over these alternatives. This model can be represented as follows. For every $i \in N$, $X_i = A$, and $\mathcal{A} = \{(a, \dots, a) \in \times_{i \in N} X_i : a \in A\}$. Then it must be that for all $i, j \in N$ and all $\alpha \in \mathcal{A}$, $\alpha_i = \alpha_j$.

For such $\mathcal{A}$, note that every rule $\varphi : \mathcal{P} \to \mathcal{A}$ is trivially group non-bossy, since preferences are strict not only over own outcomes, but also over allocations. Thus, the following corollary to Proposition 1 is immediate.

Corollary 1. *Let $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ be a Cartesian domain of preference profiles for a social choice model, and let φ be a rule on $\mathcal{P}$. Then the following statements are equivalent:*

1. *φ is strategy-proof.*
2. *φ is effectively pairwise strategy-proof.*

With the assumption of two-point connectedness on the domain, Maskin monotonicity implies strategy-proofness, by Proposition 6, and strategy-proofness implies group strategy-proofness, by Proposition 7, yielding the following corollary, a counterpart to Theorem 1 for social choice models. The well-known equivalence of strategy-proofness and Maskin monotonicity on the entire strict preference domain (Muller and Satterthwaite, 1977) follows from it.

Corollary 2. *Let $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ be a Cartesian domain of preference profiles for a social choice model, and let φ be a rule on $\mathcal{P}$. If $\mathcal{P}$ is two-point connected, then the following statements are equivalent:*

1. *φ is strategy-proof.*
2. *φ is (effectively) pairwise strategy-proof.*
3. *φ is (weakly) group strategy-proof.*[23]
4. *φ is Maskin monotonic.*

The assumption of two-point connectedness is important for the equivalence in Corollary 2. The following simple example shows that its failure allows for rules that are not group or pairwise strategy-proof, yet effectively pairwise strategy-proof and Maskin monotonic.

Example 6. *Let $N = \{i, j\}$, $X_i = X_j = \{a, b, c, d\}$, and $\mathcal{A} = \{\alpha, \beta, \delta, \gamma\}$, where $\alpha = (a, a)$, $\beta = (b, b)$, $\gamma = (c, c)$, and $\delta = (d, d)$. Let $\mathcal{P}_i = \mathcal{P}_j = \{R^1, R^2, R^3\}$ satisfy:*

R^1	R^2	R^3
a	d	d
b	c	b
c	b	c
d	a	a

Consider the rule φ on $\mathcal{P}$ defined as follows:

R^3	γ	δ	δ
R^2	γ	δ	δ
R^1	α	β	β
φ	R^1	R^2	R^3

[23]Note that for this model with no private goods and strict preferences, weak group strategy-proofness is equivalent to group strategy-proofness.

This rule is strategy-proof (and group non-bossy, and so Maskin monotonic by Propositions 1 and 2).[24] *However,* $(\{i,j\},(R^3,R^1),(R^1,R^3))$ *is a non-robust manipulation. Thus* φ *is not pairwise strategy-proof (and hence not group strategy-proof).*

Note that $\mathcal{P}_i$ *is not two-point connected — even though* $b\ P^1\ c$*, there does not exist* $\hat{R} \in \mathcal{P}_i$ *such that* $\hat{R} \in MT_i(R^1,c) \cap MT_i(R^3,b)$*.*

Single-peaked domains Let R_i be a complete, reflexive, and transitive binary relation on X_i that is not necessarily antisymmetric. Suppose X_i carries a linear order $\leq$. Preference relation R_i is *single-peaked* (on X_i with linear order $\leq$) if there exists $p_i \in X_i$ such that for every $y,z \in X_i \setminus \{p_i\}$, *a*) $y \leq p_i$ or $p_i \leq y$ implies $p_i\ P_i\ y$ and *a*) $y \neq z$ and ($z \leq y \leq p_i$ or $p_i \leq y \leq z$) implies $p_i\ P_i\ y\ P_i\ z$. Preference relation R_i in domain $\tilde{\mathcal{P}}_i$ is *strict single-peaked* (on X_i with order $\leq$) if there exists $p_i \in X_i$ such that for every $y,z \in X_i \setminus \{p_i\}$, $z < y < p_i$ or $p_i < y < z$ implies $p_i\ P_i\ y\ P_i\ z$. Let $\tilde{\mathcal{P}}_i^{SP}$ be the set of all strict single-peaked preferences on $(X_i,\leq)$ from $\tilde{\mathcal{P}}_i$. Denote the peak of $R_i \in \tilde{\mathcal{P}}_i^{SP}$ by $p(R_i)$.

Group strategy-proofness of well-known rules in settings with single-peaked preferences also follows from results in this paper. For social choice settings with strict preferences, Corollary 2 shows two-point connectedness of preference domains suffices for strategy-proofness to imply group strategy-proofness. For the model of Moulin (1980) with an ordered set of discrete social alternatives, the set of all strict single-peaked preferences is two-point connected (and even rich). Moulin (1980) characterizes the class of generalized median rules by strategy-proofness, efficiency, and anonymity. By Corollary 2, they are also group strategy-proof. In the case of private goods, consider the division problem with single-peaked preferences and a perfectly divisible good. Sprumont (1991) characterizes the uniform rule by strategy-proofness, efficiency, and anonymity, and also shows that it is weakly group strategy-proof. Suppose the unit is discretized into a countable field.[25] The set of all strict single-peaked preferences over own outcomes in an allocation constitute a rich domain. It is straightforward to show that the uniform rule is non-bossy, and so by Theorem 1 and Proposition 1, it is also group strategy-proof for the discretized problem. These two results for *continuous* single-peaked domains have been established in earlier literature. In particular, Moulin (2017) demonstrates group strategy-proofness of a large class of rules for one-dimensional settings with continuous single-peaked preferences that include generalized median rules and the uniform rule. Instead, a contribution of this paper is to show group strategy-proofness of these familiar rules is a corollary of a result for a broad class of problems beyond just single-peaked

[24] It also satisfies unanimity and has full range, but is not dictatorial.

[25] One could only allow for allocations that divide the good across agents in rational quantities, for instance.

ones. Moreover, for interesting subdomains of the strict single-peaked domain, defined next, group strategy-proofness of a rule is no more powerful a requirement than pairwise strategy-proofness, by Proposition 10 below.

To describe such domains, I first define some terms. Suppose that X_i carries a linear ordering $\leq$. Given two elements $x, y \in X_i$, a *closed interval from x to y*, denoted $[x,y]$, is the set $\{z \in X_i : x \leq z \leq y\}$. This interval is nonempty if and only if $x \leq y$. A closed interval is *non-trivial* if it is equivalent to $[x,y]$ for some $x, y \in X_i$ satisfying $x < y$. A set $Y \subseteq X_i$ is *convex* if for every $x, y \in Y$, $[x,y] \subseteq Y$. Given $Y \subseteq X_i$, define the *convex hull* of Y as $conv(Y) = \bigcup_{x,y \in Y}[x,y]$. Given convex $Z \subseteq X_i$, a subset Y of X_i is *interval-dense in Z* if every subset of Z that is a non-trivial closed interval contains an element of Y, i.e. for every $x, y \in Z$, if $x < y$, then $[x,y] \cap Y \neq \emptyset$.

I now describe an economically interesting class of preference domains for social choice models that satisfy two-point connectedness, but not necessarily richness. A strict single-peaked preference $R_i \in \tilde{\mathcal{P}}_i^{SP}$ with peak at p_i is *left-biased* if for every $y, z \in X_i \setminus \{p\}$, $y < p < z$ implies $y \; P_i \; z$.[26] A preference domain $\mathcal{P}_i \subseteq \tilde{\mathcal{P}}_i^{SP}$ is a *left-biased strict single-peaked (LB-SP)* domain if it only consists of such preferences.[27] Notice that an LB-SP preference is completely identified by its peak. A domain of LB-SP preferences $\mathcal{P}_i$ can be identified by the set of peaks of preferences in $\mathcal{P}_i$; denote this set by $\Pi_i(\mathcal{P}_i)$. An LB-SP preference domain $\mathcal{P}_i$ is *peak-dense* if the set of peaks $\Pi_i(\mathcal{P}_i)$ is interval-dense in $conv(\Pi(\mathcal{P}_i))$. It entails the following: for any two distinct alternatives in $conv(\Pi(\mathcal{P}_i))$, there exists some preference $R_i \in \mathcal{P}_i$ that has its peak in between the two alternatives (or is one of the two alternatives). An LB-SP preference domain $\mathcal{P}_i$ is *peak-convex* if $\Pi(\mathcal{P}_i) = conv(\Pi(\mathcal{P}_i))$, i.e. $\Pi(\mathcal{P}_i)$ is convex. Note that a peak-convex LB-SP domain is peak-dense, but the converse need not hold if $|X_i| \geq 3$. An LB-SP preference domain $\mathcal{P}_i$ is *essentially complete* if $\Pi(\mathcal{P}_i) = X_i$ or $\Pi(\mathcal{P}_i) = X_i \setminus \{\bar{x}\}$, where $\bar{x} \in X_i$ such that for every $y \in X_i$, $y \leq \bar{x}$. Note that an essentially complete LB-SP domain is peak-convex, but the converse need not hold if $|X_i| \geq 3$.

From the following proposition, a two-point connected LB-SP domain is characterized by peak-denseness, while a rich LB-SP domain is characterized by the stronger condition of peak-convexity. Interestingly, a one-point connected LB-SP domain is actually rich, and so two-point connected, as I show in the proof of the proposition. Condition R1 of Klaus and Bochet (2013) is defined as follows: for every $R_i \in \mathcal{P}_i$ and $x, y \in X_i$ such that $x \; P_i \; y$, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $\hat{R}_i \in MT_i(R_i, y)$ and $x \; \hat{R}_i \; z$ for every $z \in X_i$. An LB-SP domain that satisfies Condition R1 is characterized by the yet stronger requirement

[26] Right-biased strict single-peaked preferences can be defined analogously.

[27] Klaus and Bochet (2013) call such preferences "left-right single-peaked", and offer another economic example where they arise.

that the set of peaks contains every alternative in X_i, except possibly the right-most alternative.[28] So there are at most two LB-SP domains that satisfy Condition R1: the set of all LB-SP preferences, and in the case where there is a right-most element, the set of all LB-SP preferences with a peak other than the right-most element.

Proposition 10. *Let $\mathcal{P}_i$ be an LB-SP preference domain defined on $(X_i, \leq)$.*

1. *Domain $\mathcal{P}_i$ is two-point connected if and only if it is peak-dense.*
2. *Domain $\mathcal{P}_i$ is rich if and only if it is peak-convex.*
3. *Let $|X_i| \geq 3$. Domain $\mathcal{P}_i$ is satisfies Condition R1 if and only if it is essentially complete.*

An immediate consequence of Proposition 10 and Corollary 2 is that for every rule on a peak-dense LB-SP preference domain in a social choice setting, strategy-proofness is equivalent to Maskin monotonicity (and to group strategy-proofness).[29] Consider the following application of the proposition to a stylized problem of locating a public good where two-point connectedness, but not richness or Condition R1, is satisfied.

Example 7. *A retirement community wishes to organize an evening bingo game. There is a main road through the community serviced by a bus running a round-trip route that goes from east to west in the early evening, and returns west to east in the late evening, straddling game time. The community is divided into alternating blocks of residential and commercial zones. Citizens, who reside in residential zones, prefer a bus ride to a walk no matter the distances involved, and holding method of transportation fixed, prefer to spend less time commuting. The bingo game can be organized in either a residential or a commercial block.*[30] *A citizen's preferences over the possible locations for the club are strict single-peaked, with peak at one of the residential blocks and nearer blocks preferred to farther ones along one of the directions. Moreover, each block west of home is preferred to each block east of home, since a bus can be used to get to and from the game in the first, but not the second, case. So, by Proposition 10, the set of all possible preferences over the location for bingo is a two-point connected domain, but is not rich (and does not satisfy Condition R1) except in the trivial case where there is only one residential and one commercial block. Therefore, by Corollary 2, any strategy-proof rule for this problem is group strategy-proof.*

[28] The right-most alternative of X_i, denoted $\bar{x}$, must satisfy $y \leq \bar{x}$ for every $y \in X_i$. If $\bar{x}$ exists, it is unique.

[29] Klaus and Bochet (2013) shows this equivalence only for the domain of all LB-SP preferences.

[30] This can be formally modeled as follows. Let N be the finite set of citizens, and for each $i \in N$, X_i is equal to A, the finite set of all blocks on the main road, with order increasing from west to east. For instance, A could be modeled as a consecutive subset of natural numbers, where odd numbers represent the residential blocks, and even ones the commercial blocks.

Indivisible object allocation For the school choice model (Abdulkadiroğlu and Sönmez, 2003), the student-optimal stable rule is strategy-proof, but is not generally group strategy-proof. Pathak and Sönmez (2008) discuss how some parents in Boston organized meetings to share insight into how to manipulate the original rule used (the so-called Boston mechanism). While their recommendation of the agent-optimal stable rule removes any individual incentive to misreport preferences, such meetings between parents raise the possibility of joint manipulations by small groups of parents. A market designer concerned with manipulations by small groups might naturally seek a rule that is both stable and pairwise strategy-proof. From Theorem 1, we learn that any such rule is also group strategy-proof. But a stable rule is group strategy-proof if and only if the priority structure is acyclic (Ergin, 2002).

Corollary 3. *In school choice, a rule is pairwise strategy-proof and stable if and only if it is the student-optimal stable rule under an Ergin-acyclic priority structure.*[31,32]

Whenever the student-optimal stable rule is not pairwise strategy-proof, there is a preference profile at which a pair of agents robustly manipulate it. Since it is weakly group strategy-proof (Hatfield and Kojima, 2009), at this robust manipulation only one agent strictly benefits. However, by robustness, neither agent has any reason to deviate from an agreement to manipulate, so with the possibility of transfers outside the rule, both agents can be made strictly better off in a robust manner.

For the indivisible object allocation model with unit demand and strict preferences, Pycia and Ünver (2017) characterize a class of rules they call trading cycles rule by group strategy-proofness and Pareto-efficiency. Theorem 1 yields the following corollary given their characterization.[33]

Corollary 4. *In indivisible object allocation with unit demand and supply and strict preferences, a rule is pairwise strategy-proof and Pareto-efficient if and only if it is a trading cycles rule.*

For the matching with contracts model and its various applications to real-world mechanism design, agents have strict preferences over contracts involving them.[34] If all unit-demand preferences are allowed, then the richness condition is satisfied, and so

[31] This also relies on the result that the student-optimal stable rule is the unique strategy-proof and stable rule (Alcalde and Barberà, 1994).

[32] Kumano (2009) generalizes the results of Ergin (2002) to the agent-optimal stable rule in a model with choice functions that are substitutable and acceptant up to a fixed quota. A similar corollary applies to this rule given his findings.

[33] A similar corollary can be obtained for the characterization in Pápai (2000).

[34] See the applications in Sönmez and Switzer (2013) and Sönmez (2013), for instance.

the results in this paper apply. In particular, the cumulative offer mechanism (Hatfield and Milgrom, 2005) is pairwise strategy-proof if and only if it is group strategy-proof.[35]

5 Conclusion

In this paper, I have demonstrated that the combination of two-point and one-point connectedness of the preference domain, satisfied by a number of interesting problems, renders any rule that is immune to pairwise manipulations also immune to arbitrary group manipulations. On one hand, the task of identifying whether a rule has a manipulation involving more than one agent is made simpler, since only pairwise manipulations need to be checked. On the other hand, to the extent that group strategy-proofness is difficult to obtain, I offer a more negative assessment: even a slight strengthening of strategy-proofness to pairwise strategy-proofness leads to the same difficulty on such domains.

Going beyond manipulating pairs, I study manipulation by groups up to a fixed size $m \in \mathbb{Z}_+$ in Appendix A. The counterpart to pairwise strategy-proofness is m-strategy-proofness, the immunity to manipulations up to size m.[36] When the duo of domain conditions is satisfied, this hierarchy of strategy-proofness conditions amounts to pairwise strategy-proofness, leaving only individual strategy-proofness distinct. There are two notable results when one or both conditions fails to hold. The first, requiring two-point connectedness, is a version of Proposition 7 with the group size bounded by m. The second is the connection between manipulations and robust manipulations in general. If all manipulations of up to size m are precluded, then no robust manipulation by $m+1$ agents is possible either.

The framework accommodates many well-known models, such as discrete goods allocation with or without priorities, and voting models on a discrete space of ordered alternatives with single-peaked preferences. A limitation of the framework, however, is that preferences over own outcomes must be strict, which rules out problems where agents have continuous preferences over divisible goods. Recently, Barberà et al. (2016) have studied conditions under which strategy-proofness implies weak group strategy-proofness in such a setting. The relationship between manipulability by small groups and that by larger groups when there may be indifferences in outcomes and only some manipulators need to be made strictly better off remains to be thoroughly examined.

[35] I am not aware of a study on the question of group strategy-proofness of the cumulative offer mechanism. On weak group strategy-proofness, Hatfield and Kojima (2009) identify some sufficient conditions.

[36] I similarly parametrize effective pairwise strategy-proofness, non-bossiness, and Maskin monotonicity, and study their relations with each other for fixed values of m.

A Eliminating manipulations by groups of size m

I begin by generalizing previous definitions for groups of size $m \in \mathbb{Z}_+$.

A rule φ is m-**strategy-proof** if there does not exist a set $M \subseteq N$, $|M| \leq m \in \mathbb{Z}_+$, and $R, R' \in \mathcal{P}$ such that (M, R, R') is a manipulation. A rule φ is **effectively** m-**strategy-proof** if there does not exist a robust manipulation (M, R, R') with $|M| \leq m \in \mathbb{Z}_+$. A rule φ is m-**non-bossy** if for every profile $R \in \mathcal{P}$, for every set of agents $M \subseteq N$ with $|M| \leq m \in \mathbb{Z}_+$, and every preference profile $\tilde{R}_M \in \mathcal{P}_M$, if $\varphi_i(R) = \varphi_i(\tilde{R}_M, R_{-M})$ for every $i \in M$, then $\varphi(R) = \varphi(\tilde{R}_M, R_{-M})$. A rule φ is m-**monotonic** if for every $R \in \mathcal{P}$, every $M \subseteq N$ with $|M| \leq m \in \mathbb{Z}_+$, and every $R'_M \in \mathcal{P}_M$, if R'_i monotonic transformation of R_i at $\varphi_i(R)$ for every $i \in M$, then $\varphi_M(R') = \varphi_M(R)$. If $m = 1$ in this definition, the φ is **individually monotonic**.

For what follows, suppose φ is a rule defined on a Cartesian domain $\mathcal{P} \subseteq \tilde{\mathcal{P}}$, and suppose $m \in \mathbb{Z}_+$.

The following two lemmas will prove useful in the subsequent proofs. Takamiya (2001) proves the first one for a special case of my framework, and Klaus and Bochet (2013) prove both for a more general domain of preferences that allows some indifferences (their Condition R2). I include proofs for the sake of completeness.

Lemma 1 (Monotonicity Lemma). *If φ is strategy-proof, then it is individually monotonic.*

Proof. Let $R \in \mathcal{P}$ and $i \in N$, and suppose $\hat{R}_i \in \mathcal{P}_i$ is a monotonic transformation of R_i at $\varphi_i(R)$. By strategy-proofness, $\varphi_i(R)\ R_i\ \varphi_i(\hat{R}_i, R_{-i})$. Then, by the definition of a monotonic transformation, $\varphi_i(R)\ \hat{R}_i\ \varphi_i(\hat{R}_i, R_{-i})$. By strategy-proofness, $\varphi_i(\hat{R}_i, R_{-i})\ \hat{R}_i\ \varphi_i(R)$. Thus, $\varphi_i(\hat{R}_i, R_{-i})\ \hat{I}_i\ \varphi(R)$. By strict preferences, $\varphi_i(\hat{R}_i, R_{-i}) = \varphi_i(R)$. □

Lemma 2. *Let $\mathcal{P}$ be two-point connected. If φ is individually monotonic, then it is strategy-proof.*

Proof. Suppose φ is individually monotonic but not strategy-proof. I will obtain a contradiction. Since φ is not strategy-proof, there exists $i \in N$ such that $(\{i\}, R, R')$ is a manipulation for some $R, R' \in \mathcal{P}$. Then $\varphi_i(R')\ P_i\ \varphi_i(R)$. By two-point connectedness, there exists $\hat{R}_i \in \mathcal{P}$ such that $\hat{R}_i \in MT_i(R_i, \varphi_i(R)) \cap MT_i(R'_i, \varphi_i(R'_i))$. By individual monotonicity, $\varphi_i(\hat{R}_i, R'_{-i}) = \varphi_i(R')$ and $\varphi_i(\hat{R}_i, R_{-i}) = \varphi_i(R)$. Thus, $\varphi_i(R') = \varphi_i(R)$, a contradiction. □

Lemma 3. *If φ is individually monotonic and non-bossy, then it is Maskin monotonic.*

Proof. Let $R \in \mathcal{P}$ and let $\hat{R} \in \mathcal{P}$ be a monotonic transformation of R at $\varphi(R)$. Then, for each $k \in N$, $\hat{R}_k$ is a monotonic transformation of R_k at $\varphi_k(R)$. Identify each agent in N with an integer in $\{1, \ldots, |N|\}$. For each $k \in \{1, \ldots, |N|\}$, define $\hat{R}^{\leq k} = (\hat{R}_1, \ldots, \hat{R}_k, R_{k+1}, \ldots, R_{|N|})$.

Identify $\hat{R}^{\leq 0}$ with R, and notice that $\hat{R}^{\leq |N|} = \hat{R}$. Since $\mathcal{P}$ is a Cartesian product, $R, \hat{R} \in \mathcal{P}$ implies $\hat{R}^{\leq k} \in \mathcal{P}$ for every $k \in \{1, \ldots, |N|\}$.

I will show $\varphi(\hat{R}) = \varphi(R)$ by induction on the sequence of profiles $(\hat{R}^{\leq k})_{k=0}^{k=|N|}$. Trivially, $\varphi(\hat{R}^{\leq 0}) = \varphi(R)$, establishing the base case. Now, suppose $\varphi(\hat{R}^{\leq k-1}) = \varphi(R)$, where $k \in \{1, \ldots, |N|-1\}$. By individual monotonicity, $\varphi_k(\hat{R}^{\leq k}) = \varphi_k(\hat{R}^{\leq k-1})$. By non-bossiness, $\varphi(\hat{R}^{\leq k}) = \varphi(\hat{R}^{\leq k-1})$. Thus, $\varphi(\hat{R}^{\leq k}) = \varphi(R)$, completing the induction step. $\square$

Lemma 4. *Let $m \geq 1$. If φ is $(m+1)$-strategy-proof, then it is m-non-bossy.*

Proof. Let $S \subseteq N$ such that $0 < |S| \leq m$. Let $R, R' \in \mathcal{P}$ be preference profiles such that $R'_{-S} = R_{-S}$ and such that for every $i \in S$, $\varphi_i(R') = \varphi_i(R)$.

Suppose there exists $j \in N \setminus S$ such that $\varphi_j(R') \neq \varphi_j(R)$. Since preferences are strict, either $\varphi_j(R')\ P_j\ \varphi(R)$ or $\varphi_j(R)\ P_j\ \varphi(R')$. In the former case, $(S \cup \{j\}, R, R')$ constitutes a manipulation, while in the latter case, $(S \cup \{j\}, R', R)$ constitutes a manipulation. In both cases, $|S \cup \{j\}| \leq m+1$, contradicting the assumption that φ is $m+1$-strategy-proof. $\square$

Note that the group manipulation in Example 3 is not robust. The following lemma shows that this is not a coincidence. Robust manipulations of a particular size require the existence of a possibly non-robust manipulation with one less agent.

Lemma 5. *Let $m \geq 1$. If φ is m-strategy-proof and non-bossy, then it is effectively $(m+1)$-strategy-proof.*

Proof. Let (M, R, R') be a robust manipulation of φ, with $|M| \leq m+1$. By m-strategy-proofness, $|M| > m$, so $|M| = m+1$. I will prove the lemma by contradiction.

There exists $i \in M$ such that $R'_i \neq R_i$. By strategy-proofness, $\varphi_i(R_i, R'_{-i})\ R_i\ \varphi_i(R')$, but by robustness, $\varphi_i(R')\ R_i\ \varphi_i(R_i, R'_{-i})$, so by strict preferences, $\varphi_i(R_i, R'_{-i}) = \varphi_i(R')$. Non-bossiness implies $\varphi(R_i, R'_{-i}) = \varphi(R')$. Now, suppose there exists $j \in M \setminus \{i\}$ such that $\varphi_j(R')\ P_j\ \varphi_j(R)$. Then, since $\varphi(R_i, R'_{-i}) = \varphi(R')$, $(M \setminus \{i\}, R, (R_i, R'_{-i}))$ constitutes a manipulation of φ by m agents, contradicting m-strategy-proofness. Thus, given strict preferences, for every $j \in M \setminus \{i\}$, $\varphi_j(R') = \varphi_j(R)$, and $\varphi_i(R')\ P_i\ \varphi_i(R)$.

The argument in the second paragraph applies for every agent $k \in M$ such that $R'_k \neq R_k$. Therefore, to avoid a contradiction, there can be only one agent $i \in M$ such that $R'_i \neq R_i$, implying $R' = (R'_i, R_{-i})$ and for this agent $\varphi_i(R')\ P_i\ \varphi_i(R)$. But then $(\{i\}, R, R')$ is a manipulation by agent i, contradicting strategy-proofness. $\square$

Corollary 5. *Let $m \geq 2$. If φ is m-strategy-proof, then it is effectively $(m+1)$-strategy-proof.*

Proof. For $m \geq 2$, m-strategy-proofness implies non-bossiness, by Lemma 4. Thus, the result follows by Lemma 5. $\square$

The rule in Example 3 is on a two-point connected domain and is effectively 3-strategy-proof and even weakly group strategy-proof. However, it is not 3-strategy-proof (and also fails to be 2-non-bossy).

The following lemma makes clear why, in Proposition 7, group non-bossiness on the rule can take the place of one-point connectedness on the domain.

Lemma 6. *Let $\mathcal{P}$ be one-point connected and let $m \geq 1$. If φ is individually monotonic and non-bossy, then it is m-non-bossy.*

Proof. I will prove this by induction. The case for $m = 1$ is trivial, so suppose $m > 1$. Now suppose for every $m' < m$, φ is m'-non-bossy.

Let $S \subseteq N$ such that $|S| = m$. Let $R, R' \in \mathcal{P}$ be preference profiles such that $R'_{-S} = R_{-S}$ and such that for every $i \in S$, $\varphi_i(R') = \varphi_i(R)$.

Let $i \in S$. By one-point connectedness, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $\hat{R}_i \in MT_i(R_i, \varphi_i(R)) \cap MT_i(R'_i, \varphi_i(R))$. By individual monotonicity, $\varphi_i(\hat{R}_i, R_{-i}) = \varphi_i(R)$ and $\varphi_i(\hat{R}_i, R'_{-i}) = \varphi_i(R')$. Then, by non-bossiness, $\varphi(\hat{R}_i, R_{-i}) = \varphi(R)$ and $\varphi(\hat{R}_i, R'_{-i}) = \varphi(R')$, and so for every $j \in S \setminus \{i\}$, $\varphi_j(\hat{R}_i, R_{-i}) = \varphi_j(\hat{R}_i, R'_{-i})$. By the induction hypothesis, φ is $(m-1)$-non-bossy. Thus, $\varphi(\hat{R}_i, R_{-i}) = \varphi(\hat{R}_i, R'_{-i})$, and so $\varphi(R) = \varphi(R')$, concluding the induction step. □

Lemma 7. *Let $\mathcal{P}$ be one-point connected, and let $m \geq 2$. If φ is m-monotonic, then it is non-bossy.*

Proof. Let $R \in \mathcal{P}$, $i \in N$, and $R'_i \in \mathcal{P}_i$ such that $\varphi_i(R'_i, R_{-i}) = \varphi_i(R)$. By one-point connectedness, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $\hat{R}_i \in MT_i(R_i, \varphi_i(R)) \cap MT_i(R'_i, \varphi_i(R))$. For every $j \in N \setminus \{i\}$, $(\hat{R}_i, R_{-i})$ is a $\{i,j\}$-monotonic transformation of both R and (R'_i, R_{-i}) at $\varphi(R)$. Since $m \geq 2$, $\varphi_j(\hat{R}_i, R_{-i}) = \varphi_j(R)$ and $\varphi_j(\hat{R}_i, R_{-i}) = \varphi_j(R'_i, R_{-i})$, by m-monotonicity. Thus, for every $l \in N$, $\varphi_l(R'_i, R_{-i}) = \varphi_l(R)$, i.e., φ is non-bossy. □

Lemma 8. *Let $\mathcal{P}$ be two-point connected and $m \geq 1$. If φ is strategy-proof and non-bossy, then it is weakly m-strategy-proof.*

Proof. Let (M, R, R') be a strong manipulation, i.e. for every $i \in M$, $\varphi_i(R')\ P_i\ \varphi_i(R)$. I prove the lemma by induction. The case of $m = 1$ is trivial, so suppose there is no strong manipulation of size $m' < m$, where $m > 1$. Then for every $i \in M$, $R'_i \neq R_i$. Take some $i \in M$. By two-point connected, there exists $\hat{R}_i \in MT_i(R_i, \varphi_i(R)) \cap MT_i(R'_i, \varphi_i(R'))$. By the monotonicity lemma (Lemma 1), $\varphi_i(\hat{R}_i, R_{-i}) = \varphi_i(R)$ and $\varphi_i(\hat{R}_i, R'_{-i}) = \varphi_i(R')$. By non-bossiness, $\varphi(\hat{R}_i, R_{-i}) = \varphi(R)$ and $\varphi(\hat{R}_i, R'_{-i}) = \varphi(R')$. Thus, $(M \setminus \{i\}, (\hat{R}_i, R_{-i}), (\hat{R}_i, R'_{-i}))$ is a strong manipulation by $m-1$ agents, a contradiction. □

Proposition 11. *Let $\mathcal{P}$ be two-point connected and $m \geq 0$. If φ is strategy-proof and m-non-bossy, then it is $(m+1)$-strategy-proof.*

Proof. I prove the result using an induction argument over m. The statement to be proved is trivially true for the case of $m = 0$, so suppose $m > 0$. Since φ is m-non-bossy, it is m'-non-bossy for every $m' < m$, and so by the induction hypothesis and strategy-proofness, φ is m-strategy-proof.

To demonstrate $m+1$-strategy-proofness, suppose for the sake of contradiction that (M, R, R') is a manipulation by $m+1$ agents, i.e. $|M| = m+1$.

First, no manipulator strict improves while also misreporting his preferences. To see this, suppose to the contrary that such a manipulator $i \in M$ exists, i.e. $R'_i \neq R_i$ and $\varphi_i(R')\ P_i\ \varphi_i(R)$. By two-point connectedness, there exists $\hat{R}_i \in MT_i(R_i, \varphi_i(R)) \cap MT_i(R'_i, \varphi_i(R'))$. By the monotonicity lemma (Lemma 1) and strict preferences, $\varphi_i(\hat{R}_i, R_{-i}) = \varphi_i(R)$ and $\varphi_i(\hat{R}_i, R'_{-i}) = \varphi_i(R')$. By non-bossiness, $\varphi(\hat{R}_i, R_{-i}) = \varphi(R)$ and $\varphi(\hat{R}_i, R'_{-i}) = \varphi(R')$. Thus, $(M, (\hat{R}_i, R_{-i}), (\hat{R}_i, R'_{-i}))$ is a manipulation by the same $m+1$ agents with the same pattern of welfare improvements for manipulators, albeit one where agent i strictly improves without misreporting his preferences. Since $m > 0$, there is at least one other agent in M. Suppose another manipulator $j \in M \setminus \{i\}$ also strict benefits from the manipulation $(M, (\hat{R}_i, R_{-i}), (\hat{R}_i, R'_{-i}))$, i.e. $\varphi_j(\hat{R}_i, R'_{-i})\ P_j\ \varphi_j(\hat{R}_i, R_{-i})$. Then $(M \setminus \{i\}, (\hat{R}_i, R_{-i}), (\hat{R}_i, R'_{-i}))$ is also a manipulation, but by m agents, which contradicts m-strategy-proofness. So, no manipulator other than i strictly improves through that manipulation, i.e. for every $j \in M \setminus \{i\}$, $\varphi_j(\hat{R}_i, R_{-i})\ I_j\ \varphi_j(\hat{R}_i, R'_{-i})$. By strict preferences, $\varphi_j(\hat{R}_i, R_{-i}) = \varphi_j(\hat{R}_i, R'_{-i})$. This in turn implies $\varphi(\hat{R}_i, R_{-i}) = \varphi(\hat{R}_i, R'_{-i})$, since φ is m-non-bossy, $|M \setminus \{i\}| = m$, and $R'_k = R_k$ for every $k \in N \setminus M$. But then agent i does not strictly improve through the manipulation $(M, (\hat{R}_i, R_{-i}), (\hat{R}_i, R'_{-i}))$, contradicting a previous claim. Thus, it must be that any manipulator $i \in M$ who strictly improves in manipulation (M, R, R') does not misreport his preferences, i.e. $\varphi_i(R')\ P_i\ \varphi_i(R)$ implies $R'_i = R_i$.

Second, there is no more than one manipulator who strictly improves in the manipulation (M, R, R'). To see this, suppose to the contrary that there exist distinct $i, j \in M$ such that $\varphi_i(R')\ P_i\ \varphi_i(R)$ and $\varphi_j(R')\ P_j\ \varphi_j(R)$. Having already established that strict improvement for a manipulator implies no preference misrepresentation, $R'_j = R_j$, it must be that $(M \setminus \{j\}, R, R')$ is also be a manipulation, but by m agents. But this contradicts m-strategy-proofness.

Thus, the manipulation (M, R, R') has exactly one agent $i \in M$ who strictly improves from the manipulation, while every other agent $j \in M \setminus \{i\}$ is indifferent to the manipulation, and moreover, i does not misrepresent his preferences. That is, $R' = (R'_{M\setminus\{i\}}, R_i, R_{-M})$ and $\varphi_j(R')\ I_j\ \varphi_j(R)$, the latter of which implies $\varphi_j(R') = \varphi(R)$, by strict preferences. Then, by m-non-bossiness, $\varphi(R) = \varphi(R'_{M\setminus\{i\}}, R_i, R_{-M}) = \varphi(R')$, contradicting the claim that i strictly improves, and concluding the proof of the induction step. $\square$

B Proofs omitted from body

The proofs here rely upon the results on general manipulations in Appendix A.

Proof of Proposition 1. First, I show that an effectively pairwise strategy-proof rule is non-bossy. Strategy-proofness is of course obvious by definition. Let $i \in N$, $R, R' \in \mathcal{P}$, where $R'_j = R_j$ for every $j \in N \setminus \{i\}$. Suppose $\varphi_i(R) = \varphi_i(R')$. This implies that $\varphi_i(R')\ R_i\ \varphi_i(R)$ and $\varphi_i(R)\ R'_i\ \varphi_i(R')$. Let $j \in N \setminus \{i\}$. By effective pairwise strategy-proofness, $(\{i,j\}, R, R')$ is not a robust manipulation. Since $\varphi_i(R')\ R_i\ \varphi_i(R)$, it must be that $\varphi_j(R)\ R_j\ \varphi_j(R')$. Also by effective pairwise strategy-proofness, $(\{i,j\}, R', R)$ is not a robust manipulation. Since $\varphi_i(R)\ R'_i\ \varphi_i(R')$ and given that $R'_j = R_j$, it must be that $\varphi_j(R')\ R_j\ \varphi_j(R)$. Thus, $\varphi_j(R)\ I_j\ \varphi_j(R')$. Since R_j is antisymmetric, $\varphi_j(R) = \varphi_j(R')$.

For the other direction, the result follows from Lemma 5. □

Proof of Proposition 2. By Lemma 1, strategy-proofness implies individual monotonicity (see Appendix A for the definition of this). Then the result follows from Proposition 1, which implies non-bossiness, and Lemma 3. □

Proof of Proposition 3. Follows from the more general result of Proposition 11 in Appendix A, together with Proposition 1. □

Proof of Proposition 4. One direction is immediate.

As for the converse, given Proposition 1, if there is a non-robust pairwise manipulation both manipulating agents change their reports, though only one of them need strictly benefit from the manipulation. Otherwise non-bossiness would fail. However, each agent would strictly benefit by deviating from the manipulation and reporting his true preference, and each agent would be strictly worse off if his co-manipulator were to deviate from the agreed manipulation by reporting instead his true preference. This argument requires that at least four outcomes exist for one of the agents, and three outcomes for the other. So if there are no more than three distinct outcomes for every agent, or if every agent but one has no more than two distinct outcomes, effective pairwise strategy-proofness implies pairwise strategy-proofness, completing the proof. □

Proof of Proposition 5. By two-point connectedness and Propositions 1 and 3, pairwise strategy-proofness implies non-bossiness (and strategy-proofness). Then Lemma 8 completes the proof. □

Proof of Proposition 6. By definition, Maskin monotonicity implies individual monotonicity (see Appendix A for the definition of this). The result follows by two-point connectedness and Lemma 2. □

Proof of Proposition 7. Follows from the more general result of Proposition 11 in Appendix A. □

Proof of Proposition 8. Follows from the more general result of Lemma 6. □

Proof of Proposition 9. Follows from the combination of Lemma 6 and Lemma 7. □

Proof of Theorem 1. By Remark 2, the combination of one- and two-point connectedness is equivalent to richness.

First, I show that statements 1), 4), 5), and 6) are equivalent. By definition, statement 5) implies 4). By Proposition 1, statement 4) is equivalent to statement 1). By Proposition 2, statement 1) implies 6). By Proposition 6 and two-point connectedness, Maskin monotonicity implies strategy-proofness. By Proposition 9 and one-point connectedness, Maskin monotonicity implies group non-bossiness. Thus, statement 6) implies 5).

Next, I show that statements 1), 2), 3), and 5) are equivalent. By definition, statement 3) implies 2), which in turn implies 1). From above, statement 1) is equivalent to 5). By Proposition 7 and two-point connectedness, statement 5) is equivalent to 3). □

Proof of Proposition 10. Let $\mathcal{P}_i$ be an LB-SP preference domain. I begin with the following claim, the straightforward proof of which I omit.

Claim 1: Let $R_i, \hat{R}_i \in \mathcal{P}_i$ and $x \in X_i$.

(a) If $p(R_i) < x$, then $\hat{R}_i \in MT_i(R_i, x)$ if and only if $p(\hat{R}_i) \leq x$.
(b) If $x \leq p(R_i)$, then $\hat{R}_i \in MT_i(R_i, x)$ if and only if $p(\hat{R}_i) \in [x, p(R_i)]$.

Repeated application of Claim 1 yields the following claim, the proof of which I omit.

Claim 2: Let $R_i, R'_i, \hat{R}_i \in \mathcal{P}_i$ and $x, y \in X_i$, where $x \; P_i \; y$.

(a) If $p(R_i) < y$ and $p(R'_i) < x$, then $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R'_i, x)$ if and only if $p(\hat{R}_i) \leq \min\{x, y\}$.
(b) If $p(R_i) < y$ and $x \leq p(R'_i)$, then $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R'_i, x)$ if and only if $p(\hat{R}_i) \in [x, \min\{y, p(R'_i)\}]$.
(c) If $y < p(R_i)$ and $p(R'_i) < x$, then $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R'_i, x)$ if and only if $p(\hat{R}_i) \in [y, x]$.
(d) If $y < p(R_i)$ and $x \leq p(R'_i)$, then $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R'_i, x)$ if and only if $p(\hat{R}_i) \in [x, \min\{p(R_i), p(R'_i)\}]$.

With Claim 2, I show that an LB-SP domain $\mathcal{P}_i$ is two-point connected if and only if it is peak-dense.

(Two-point connected implies peak-dense) Let $[x, y] \subseteq conv(\Pi(\mathcal{P}_i))$, where $x < y$. By definition of the convex hull, there exist $R_i, R'_i \in \mathcal{P}_i$ such that $p(R_i) \leq x < y \leq p(R'_i)$,

without loss of generality. Now, $p(R_i) \leq x < y$ implies xP_iy. By two-point connectedness, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R_i', x)$. Then by Claim 2(b), $p(\hat{R}_i) \in [x, \min\{y, p(R_i')\}] = [x, y]$. Since x and y are arbitrary, $\mathcal{P}_i$ is peak-dense.

(Peak-dense implies two-point connected) Let $R_i, R_i' \in \mathcal{P}_i$, and $x, y \in X_i$ such that $x \; P_i \; y$. Note that $p(R_i) \neq y$. I show that there exists $\hat{R}_i \in \mathcal{P}_i$ such that $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R_i', x)$, by considering four cases.

Case (a): $p(R_i) < y$ and $p(R_i') < x$: Let $\hat{R}_i = R_i' \in \mathcal{P}_i$. Then $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R_i', x)$, by Claim 2(a), since $p(\hat{R}_i) = p(R_i') \leq x$.

Case (b): $p(R_i) < y$ and $x \leq p(R_i')$: Note that $x < y$, since $x \; P_i \; y$. If $x \leq p(R_i)$, then let $\hat{R}_i = R_i$. Since $p(\hat{R}_i) = p(R_i) \in [x, y]$, by Claim 2(b), $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R_i', x)$. Otherwise, $p(R_i) < x$. If $p(R_i') = x$, then let $\hat{R}_i = R_i'$, so that $p(\hat{R}_i) = x$. Instead, if $x < p(R_i')$, then $[x, \min\{y, p(R_i')\}]$ is a non-trivial closed interval of $conv(\Pi(\mathcal{P}_i))$. Since $\mathcal{P}_i$ is peak-dense, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $p(\hat{R}_i) \in [x, \min\{y, p(R_i')\}]$. In either case, by Claim 2(b), $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R_i', x)$.

Case (c): $y < p(R_i)$ and $p(R_i') < x$: Note that $x \in [y, p(R_i)]$, since $x \; P_i \; y$. Then, $p(R_i') < x \leq p(R_i)$, so $[\max\{y, p(R_i')\}, \min\{p(R_i), x\}]$ is a non-trivial closed interval of $conv(\Pi(\mathcal{P}_i))$. Since $\mathcal{P}_i$ is peak-dense, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $p(\hat{R}_i) \in [\max\{y, p(R_i')\}, \min\{p(R_i), x\}] \subseteq [y, x]$. So, by Claim 2(c), $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R_i', x)$.

Case (d): $y < p(R_i)$ and $x \leq p(R_i')$: Note that $x \in [y, p(R_i)]$, since $x \; P_i \; y$. Let $\hat{R}_i = R_i$ if $p(R_i) \leq p(R_i')$ and $\hat{R}_i = R_i'$ otherwise. Then, $p(\hat{R}_i) = \min\{p(R_i), p(R_i')\} \in [x, \min\{p(R_i), p(R_i')\}]$. So, by Claim 2(d), $\hat{R}_i \in MT_i(R_i, y) \cap MT_i(R_i', x)$.

Next, I show that an LB-SP domain $\mathcal{P}_i$ is one-point connected if and only if it is peak-convex. Since a peak-convex $\mathcal{P}_i$ is peak-dense, it is also two-point connected, from above. Also, a rich LB-SP domain is one-point connected. Thus, an LB-SP domain is rich if and only if it is peak-convex.

(One-point connected implies peak-convex) Let $x \in conv(\Pi(\mathcal{P}_i))$. By definition of the convex hull, there exists $R_i, R_i' \in \mathcal{P}_i$ such that $p(R_i) \leq x \leq p(R_i')$, without loss of generality. By one-point connectedness, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $\hat{R}_i \in MT_i(R_i, x) \cap MT_i(R_i', x)$. By Claim 1, $p(\hat{R}_i) \leq x$ and $p(\hat{R}_i) \in [x, p(R_i')]$, so $p(\hat{R}_i) = x$. Since x is arbitrary, $conv(\Pi(\mathcal{P}_i)) = \Pi(\mathcal{P}_i)$.

(Peak-convex implies one-point connected) Let $R_i, R_i' \in \mathcal{P}_i$ and $x \in X_i$. Suppose $p(R_i) \leq p(R_i')$, without loss of generality. If $x \in [p(R_i), p(R_i')]$, then by peak-convexity there exists $\hat{R}_i \in \mathcal{P}_i$ such that $p(\hat{R}_i) = x$. Then it follows from Claim 1 that $\hat{R}_i \in MT_i(R_i, x) \cap MT_i(R_i', x)$. If $x \leq p(R_i)$, let $\hat{R}_i = R_i \in \mathcal{P}_i$. Then $x \leq p(\hat{R}_i) = p(R_i) \leq p(R_i')$, so by Claim 1, $\hat{R}_i \in MT_i(R_i, x) \cap MT_i(R_i', x)$. Finally, if $p(R_i') \leq x$, let $\hat{R}_i = R_i' \in \mathcal{P}_i$. Then $p(R_i) \leq p(R_i') = p(\hat{R}_i) \leq x$, so by Claim 1, $\hat{R}_i \in MT_i(R_i, x) \cap MT_i(R_i', x)$.

Finally, I show that an LB-SP domain $\mathcal{P}_i$ satisfies Condition R1 if and only if it is essentially complete.

$\Pi(\mathcal{P}_i) = X_i$ or $\Pi(\mathcal{P}_i) = X_i \setminus \{\bar{x}\}$, where $\bar{x} \in X_i$ satisfies $y \leq \bar{x}$ for every $y \in X_i$.

(Condition R1 implies essentially complete) Let $x, y \in X_i$ such that $x < y$ and suppose $R_i \in \mathcal{P}_i$ such that $p(R_i) < y$. Then since R_i is LB-SP, $x\ P_i\ y$, so by Condition R1, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $p(\hat{R}_i) = x$ and $\hat{R}_i \in MT_i(R_i, y)$. Thus, when there exists $R_i \in \mathcal{P}_i$ such that $p(R_i)$ is not the right-most alternative, $x \in \Pi(\mathcal{P}_i)$ for every $x \in X_i$ such that x is not the right-most alternative. Instead, suppose $R_i \in \mathcal{P}_i$ such that $p(R_i)$ is the right-most alternative. Since $|X_i| \geq 3$, there exist $x, y \in X_i$ such that $x < y < p(R_i)$. Then since R_i is single-peaked, $y\ P_i\ x$, so by Condition R1, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $p(\hat{R}_i) = y$ and $\hat{R}_i \in MT_i(R_i, x)$. Thus, $y \in \Pi(\mathcal{P}_i)$ for every $y \in X_i$ such that there exists $x \in X_i$ such that $x < y$. Putting it together, if $|X_i| \geq 3$, then $x \in X_i$ and $x \notin \Pi(\mathcal{P}_i)$ implies x is the right-most alternative.

(Essentially complete implies Condition R1) Let $R_i \in \mathcal{P}_i$ and $x, y \in X_i$ such that $x\ P_i\ y$. I will show that there exists $\hat{R}_i \in \mathcal{P}_i$ such that $p(\hat{R}_i) = x$ and $\hat{R}_i \in MT_i(R_i, y)$. Since $x\ P_i\ y$, note that $x < y$ if and only if $p(R_i) < y$, by definition of LB-SP preferences. Suppose x is not the right-most alternative. Then $p(R_i) < y$. Since $\mathcal{P}_i$ is essentially complete, there exists $\hat{R}_i \in \mathcal{P}_i$ such that $p(\hat{R}_i) = x < y$. Then by Claim 1(a), $\hat{R}_i \in MT_i(R_i, y)$. Instead, suppose x is the right-most alternative. In this case, by the definition of LB-SP preferences, it must be that $p(R_i) = x$. Letting $\hat{R}_i = R_i \in \mathcal{P}_i$, it is clear that $p(\hat{R}_i) = x$ and $\hat{R}_i \in MT_i(R_i, y)$. Thus, Condition R1 is satisfied. $\square$

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