Stability and Matching with Aggregate Actors: A Perspective on Markets and Hierarchies

Samson Alva[*]

Department of Economics, University of Texas at San Antonio[†]

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Abstract

Many real-life problems involve the matching of talented individuals to institutions such as firms, hospitals, or schools, where these institutions are simply treated as individual agents. In this paper, I study many-to-one matching with contracts that incorporates a theory of choice of institutions, which are aggregate actors, composed of divisions that are enjoined by an institutional governance structure (or mechanism). Conflicts over contracts between divisions of an institution are resolved by the institutional governance structure, whereas conflicts between divisions across institutions are resolved, as is typically the case, by talents' preferences.

Noting that hierarchies are a common organizational structure in institutions, I offer an explanation of this fact as an application of the model, where stability is a prerequisite for the persistence of organizational structures. I show that stable market outcomes exist whenever institutional governance is hierarchical and divisions consider contracts to be bilaterally substitutable. In contrast, when governance in institutions is non-hierarchical, stable outcomes may not exist. Since market stability does not provide an impetus for reorganization, the persistence of markets with hierarchical institutions can thus be rationalized. Hierarchies in institutions also have the attractive incentive property that in a take-it-or-leave-it bargaining game with talents making offers to institutions, the choice problem for divisions is straightforward and realized market outcomes are pairwise stable, and stable when divisions have substitutable preferences.

Keywords: matching, governance, institutions, stability, hierarchies, organizational design

1 Introduction

Hierarchies of decision-makers are the dominant form of organizational design in a wide variety of institutions, from social institutions such as families and communities, to political institutions such as the executive branch of government, to economic institutions such as large corporations or small firms. This robust empirical fact of real-world organizations has prompted many theories to explain their existence and their functioning. Given the key role firms play in the operation of the economy, the hierarchical firm is of particular interest to economists and organizational theorists. Managerial hierarchies determine the allocation of resources within the firm, particularly through their role in conflict resolution, and also enable coordination of activities in the firm. A potential alternative to hierarchies for internal allocation is a market-like exchange mechanism, where claims on resources are more widely distributed within the organization, in the manner of cooperatives. However, while firms may have lateral equity, they usually still possess a clear vertical structure[1].

Many theories have been proposed to explain the existence of hierarchies in real-world organization of production, an institution at odds with the decentralized market mechanism coordinating economic activity. The transactions costs and incomplete contracts theories and the procedural rationality theory are some responses to this limitation of the basic theory of the firm. One goal of these theories has been to explain why firms exist or why they may be hierarchical, usually taking the market as exogenous and unaffected by the organizational design of the firm. I wish, instead, to turn the question on its head and ask how the organizational design of institutions can impact the performance of the market as a whole, where the market constitutes the free environment with institutions and individuals.

In this paper I argue that the organizational structure within each institution, what I identify as its *governance structure*, can indeed have important implications for market-level outcomes and market performance. Specifically I study how *complex* institutions, each composed of multiple actors called *divisions* with varying interests mediated by an institutional governance structure, come to make market-level choices. The governance structure is a defining feature of the institution, a product of its internal rules of coordinated resource allocation, conflict resolution, and culture. A production team in a firm, for example, could demand the same skilled worker as another team, creating a conflict for the human resource. The skilled worker may have a preference for one team over another, but this preference may not be sufficient to effect a favorable institutional decision, due to a governance structure that in this case strongly empowers the less-preferred production team. Unlike the market governance structure, where parties can freely negotiate and associate, an institutional governance structure can restricts how parties inside the institution can do so.

The main result of this paper is that whenever institutions have governance structures that are

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[1]For evidence on hierarchies and decentralization in firms, their impact on productivity, see Bloom et al. (2010).

inclusive hierarchies then stable market outcomes will exist. This existence result for the aggregate actors matching model relies upon the existence result of Hatfield and Kojima (2010), who generalize the many-to-one matching with contracts model of Hatfield and Milgrom (2005). The emergent choice behavior of institutions that have inclusive hierarchies is bilaterally substitutable whenever the divisions have bilaterally substitutable choice functions. In essence, inclusive hierarchies preserve the property of bilaterally substitutability of choice, leading to the existence result. Also preserved by this aggregation procedure is the Irrelevance of Rejected Contracts condition introduced by Aygün and Sönmez (2012b), which is a maintained assumption throughout this paper. As shown by those authors in Aygün and Sönmez (2012a), this condition is required when working with choice functions rather than with preferences as primitive. Other choice properties that are preserved include the weak substitutes condition of Hatfield and Kojima (2008) and the Strong Axiom of Revealed Preference.

Many transactions in the real world have the feature that one side is an individual such as a supplier of labor or intermediate inputs and the other side is an institution such as a large buyer firm, where the individual seeks just one relationship but the institution usually seeks many with different individuals. The standard model of matching where institutional welfare matters assumes that the institution is a single-minded actor with preferences, just like the individuals on the other side, but this black-box approach does not allow for an analysis of institutional level details. In practice, institutional choice behavior is determined by multiple institutional actors within a governance structure, which is the set of rules and norms regulating the internal functioning of the institution. As institutions seek to allocate resources amongst competing internal objectives, perhaps embodied in the divisions of the institutions, they often do so often without resorting to a price mechanism, but to a hierarchical mechanism instead. A central contribution of my work is to explain this fact by analyzing the interplay between institutional governance and market governance of transactions, which in spite of being an empirical feature of many real-world markets has been relatively unstudied from the matching perspective.

I use the matching model with aggregate actors to provide a theory for the widespread presence in firms of hierarchies with partial decentralization in decision-making in the context of factor markets. I show that hierarchical firms transacting with heterogeneous individuals in a market leads to outcomes that are in the core of the economy and are stable in a matching-theoretic sense. I support this observation by showing via examples how even in a simple setting with basic contracts (where a contract only specifies the two parties involved) and with unit-demand for factors by every division within the firms, an internal governance structure that distributes power more broadly amongst divisions and allows for trading by divisions of claims to contracts can create market-level instabilities that result in non-existence of stable or core outcomes. While this example does not rule out the possibility of market stability with such internal governance structures, it does demonstrate

the difficulty of constructing a general theory in this regard while maintaining the importance of stability of market outcomes.

The importance of institutional-level analysis of choice has been amply demonstrated in the recent market design work of Sönmez and Switzer (2012), Sönmez (2011) and Kominers and Sönmez (2012). These authors study market design where the objectives of institutions can be multiple and complex, and the manner in which these objectives are introduced into the design has a material effect on design desiderata such as stability and strategyproofness. My work is similar to these authors' works in the feature that choice is realized by an institutional procedure, though in the case of market design the only agents for the purposes of welfare are the individuals. My work is also similar to Westkamp (2012), who studies a problem of matching with complex constraints using a sequential choice procedure.

This paper, and the previously mentioned work in market design, rests upon the theory of stable matchings, initiated by Gale and Shapley (1962), which has been one of the great successes of economic theory, providing an analytical framework for the study of both non-monetary transactions and transactions with non-negligible indivisibilities.[2] This theory underpins the work in market design, where solutions to real-world allocation problems cannot feature monetary transfers and centralized mechanisms can overcome limitations of a decentralized market. Matching theory is also illuminating in the study of heterogeneous labor markets and supply chain networks, where transactions between agents are conducted in a decentralized setting. The approach of studying a heterogeneous labor market using a matching-theoretic framework was pioneered by Crawford and Knoer (1981) and Kelso and Crawford (1982), and further explored by Roth (1984b) and Roth (1985). Hatfield and Milgrom (2005) provide the modern matching with contracts framework on which much new work in matching theory is built, this paper included. Ostrovsky (2008) studies supply networks using the matching with contracts approach, work that has been followed by Westkamp (2010), Hatfield and Kominers (2012b), and Hatfield et al. (2012).

The real-world relevance of stability has been part of the extensive evidence collected by Alvin Roth for the usefulness of the matching framework for understanding *inter alia* professional labor markets. In Roth (1984a), the author describes and analyzes the history of the market for medical residents in the United States, and makes the case that stability of outcomes affected the evolution of the organizational form of the market, and that the success and persistence of the National Residency Matching Program should be attributed to the stability of the outcomes it produces under

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[2]The theoretical argument that final market outcomes will be stable can be traced back to the Edgeworth's approach to realized transactions as "finalized settlements", which are "contract[s] which cannot be varied with the consent of all parties to it [and] …which cannot be varied by recontract within the field of competition" (see pg. 19 of Edgeworth (1881)). The core of a game is a generalization of Edgeworth's recontracting notion, and the stability concept of Gale and Shapley the analogue of the core for the class of two-sided matching problems, when considered in the cooperative game framework.

straightforward behavior. Further support for the relevance of stability comes from the evidence provided in Roth (1991), where the author documents a natural experiment in the use of a variety of market institutions in a number of regional British markets for physicians and surgeons. In regions with matching procedures that under straightforward behavior produce stable outcomes, the procedures were successful in making the market operate smoothly and persisted. In some regions where the procedures in use did not necessarily produce stable outcomes, the market eventually failed to work well and these procedures were abandoned.[3] While this evidence might be construed as support for centralization of matching, the market forces are unrelated to the centralization or decentralization of the market, most clear in the fact that some of the centralized regional procedures in Britain failed to survive. Instead, the evidence points to the importance of the final outcome being a stable one.

In order to provide a non-cooperative game-theoretic understanding of my model, I study a two-stage game where talents make offers to institutions in the first stage, and then divisions within institutions choose from the available set of offers by using the internal mechanism of the institution. Focusing on subgame perfect Nash equilibria, I show that with hierarchical structures these equilibria yield pairwise stable outcomes. This supports the argument for inclusive hierarchical governance structures, in this case relying upon the notion that as internal mechanisms they have good local incentive properties for a given choice situation, in addition to their market-stability properties.

The positive and normative properties of hierarchies as allocative mechanisms when modeled as dictatorial structures has been explored in the indivisible goods setting (see Sönmez and Ünver (2011) for a survey) and in the continuous setting; for a hierarchical counterpart to the classic exchange economy model, see for example Piccione and Rubinstein (2007).[4]

The closest line of inquiry, in terms of both question and method, is Demange (2004). Her work focuses on explaining hierarchies as an organizational form for a group given a variety of coordination problems facing this group, using a cooperative game approach with a characteristic function to represent the value of various coalitions. With superadditivity, she finds that hierarchies distribute blocking power in such a way that the core exists. An important difference in this paper is the presence of multiple organizations in a bigger market. My analysis complements her study in showing that hierarchies are important not only because they produce stability in her sense, but also because they behave well in competition in a bigger market.

A well-established theory of hierarchies in organizations is the transaction costs theory, intro-

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[3]The British study is all the more intriguing because of the survival of a particular class of unstable procedures. Roth (1991) suggests that the smallness of these particular markets (numbering two) might be playing a role by removing the "impersonal" aspect of the other larger markets.

[4]There are a host of papers studying non-price mechanisms, some of which can serve as models of hierarchies. Some important works include Satterthwaite, Sonnenschein (1981), Svensson (1999), Pápai (2000), Piccione and Razin (2009), and Jordan (2006).

duced by Ronald Coase in 1937 and then thoroughly pursued by Oliver Williamson (see Williamson (2002) for a more recent summary). In the transactions costs theory, not all market transactions can be secured solely through contracts, because the governance rules of the market do not allow for it. For example, the buyer of a specific input could contract with one of a number of potential suppliers, but the relationship is plagued by the problem of hold up, since the outside value of the input is low. This example of a transaction cost, it is argued, is avoided by a vertical integration of production into the buying firm.[5]

Yet another perspective on hierarchies is the procedural rationality approach of Herbert Simon, perhaps best captured by the following quotation from a lecture in his book *The New Science of Management Decision*:

> An organization will tend to assume hierarchical form whenever the task environment is complex relative to the problem-solving and communicating powers of the organization members and their tools. Hierarchy is the adaptive form for finite intelligence to assume in the face of complexity.

Simon explained how the complexity of decision problems facing large firms cannot be solved by the individual entrepreneur, as is the characteristic assumption of the neoclassical theory of the firm. Instead, the organizational response to these problem-solving difficulties is to divide decision-making tasks within the organization and use procedures to coordinate and communicate smaller decisions in the pursuit of large goals. This information processing approach has been studied by a host of researchers, especially early on by Jacob Marschak and Roy Radner.[6]

In this paper, I abstract from informational concerns with decision-making, concentrating instead on the relationship between the capabilities of coalitions and outcomes to understand what relational structures are compatible with the preferences of actors (operationalized through the notion of stability). The origins of the decision hierarchies might be multiple, but their persistence too deserves explanation.

The remainder of the paper is organized as follows. In section 2, I describe and explain the formal framework, which I then use towards a theory of hierarchical institutions in section 3, where I also foray into an larger class of institutional structures to demonstrate that hierarchies are distinguished. In section 4, I take a non-cooperative approach and study a take-it-or-leave-it bargaining game. I conclude in section 5. Some proofs are to be found in the appendix, which also contains

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[5]Hierarchies also arise in the literature on property rights and incomplete contracts, where a fundamental inability to write comprehensive contracts makes arms-length transactions less attractive in comparison to direct control. See the seminal works of Grossman and Hart (1986) and Hart and Moore (1990), and Gibbons (2005) for a survey on theories of the firm.

[6]See Radner (1992) for a survey on hierarchies with a focus on the information processing approach. Other important works in a similar vein include the communication network of Bolton and Dewatripont (1994) and the knowledge-based hierarchy of Garicano (2000).

a section on useful comparative statics of combinatorial choice in matching and a section on the relationship between stability and the weaker notion of pairwise stability.

2 A Model of Matching with Institutions

2.1 The Elements

Let I be the set of **talents** and $(D(k))_{k \in K}$ be the collection of the set of **divisions** indexed by the set K, which is the set of (local) **institutions**, where all these sets are disjoint from each other. Associated with an institution k is a **governance structure** ψ^k, which are institutional-level rules and culture that determine how transactions involving institutional members can be secured. In the background is the **market governance structure**, which is the ambient framework within which talents and institutions conduct **market transactions**. The market governance structure determines the security of transactions between talents and institutions, but is superceded by the institutional governance structure for intra-institutional transactional details.

Transactions are modeled as bilateral **contracts** that describe the parties to the transaction as well as other details of the transaction. A contract x will name one talent $I(x)$ and one institution $K(x)$. A contract might also name one or more divisions from $D(K(x))$, but this, as with other intra-institutional details, are left free. Contracts are *comprehensive* in the sense that they describe completely all talent-institution transactional matters.[7] For a given talent i and institution k, let $X(i, k)$ be the set of all possible contracts between them. Then, $X(i) \equiv \bigcup_{k \in K} X(i, k)$ is the collection of all possible contracts involving talent i, $X(k) \equiv \bigcup_{i \in I} X(i, k)$ is the collection of all possible contracts involving institution k, and $X \equiv \bigcup_{i \in I} \bigcup_{k \in K} X(i, k)$ is the contract set for this economy, which is taken to be exogenous.

An institution transacts with potentially multiple talents in pursuit of its goals, but a talent transacts with at most one institution. Let $\mathcal{X}(i)$ be the collection of subsets of $X(i)$ that are **feasible** for i, where the empty set $\emptyset$, representing the outside option (being unmatched) for i, is always assumed to be feasible. In keeping with the assumption that a talent can have at most one contract with *any* institution, it must be that for any $Y \in \mathcal{X}(i)$, $|X(i) \cap Y| \leq 1$. We will identify these singleton sets with the element they contain for notational convenience.[8] If $|X(i) \cap X(k)| = 1$ for all $i \in I$ and $k \in K$, then the contract set is **classical**.

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[7]To the extent that a contract encodes all the details of a relationship that matter to either party, and that the set of contracts allows for every combination that could matter, this assumption is innocuous.

[8] A brief description of notation is in order. An arbitrary map f from domain E to codomain F associates each element $e \in E$ with a subset $f(e) \subseteq F$ of the codomain i.e. it is a correspondence. If for all $e \in E$, $|f(e)| = 1$, then f is a function. I will use maps from a set to some other set (where typically one of these two sets is a subset of X) to work with the relational information encoded in contracts, using the symbol for the target set as the symbol for the mapping as well. So, for any $x \in X$, $I(x)$ is the subset of talents associated with contract x, and $K(x)$ the subset of institutions.

Each talent i has strict preferences[9] $\mathcal{P}^i$ over the set of feasible contracts $\mathcal{X}(i)$ naming him. Let $\mathcal{R}^i$ be the associated weak preference relation, where $Y\mathcal{R}^iY'$ if $Y\mathcal{P}^iY'$ or $Y = Y'$. The choice behavior in a given **choice situation** $Y \subseteq X(i)$ for a talent i is determined by the preferences of this talent.[10] Let C^i denote the choice function of talent i, defined for every possible choice situation $Y \subseteq X(i)$, such that $C^i(Y) \subseteq Y$ is feasible. The assumption of preference maximization is that $C^i(Y)\mathcal{R}^iZ$ for all feasible $Z \subseteq Y$. Strict preferences implies that the maximizer is unique and thus that choice *functions* are appropriate.

In keeping with the purpose of building a model of market behavior of the institution, we will focus on the choice behavior of the institution with respect to contracts with talents. A **choice situation** for k is a subset of contracts $Y \subseteq X(k)$, a set of potential transactions that is available to the institution. Because institutions are *complex* entities, composed of many divisions with various interests, the choice behavior of an institution is an *emergent* phenomenon, shaped by the institutional governance structure ψ^k that mediate the interests of these divisions. The ideal choice of the institution in a given choice situation Y is a feasible subset $C \subseteq Y$. But whence choice?

Towards the goal of understanding institutional choice within the cooperative framework, I recognize the workings of the institution depend on the particulars of its governance structure and the interests of its actors, with the choice behavior of the institution in the market being thereby determined. I model this theory of the institution as follows: for every division $d \in D(k)$, there is an associated **domain of interest** $X(d) \subseteq X(k)$ (domains of interest of different divisions may overlap). A division d has strict preferences $\mathcal{P}^d$ over subsets of contracts in its domain of interest $X(d)$. Fixing the collection of domains of interest $\mathcal{D}(k) \equiv \{X(d)\}_{d\in D(k)}$ and the preferences of the divisions $\mathcal{P}(k) \equiv \{\mathcal{P}^d\}_{d\in D(k)}$, the institutional governance structure ψ^k determines for every choice situation $Y \subseteq X(k)$ the choice of the institution. Let C^k be the institution's **derived choice**, where the dependence on ψ^k, $\mathcal{D}(k)$, and $\mathcal{P}(k)$ has been suppressed. Choice behavior of an institution does not necessarily arise from the preference maximization of a single preference relation, unlike a talent. To the extent that a profit function can be modeled as the preference relation of a firm, the neoclassical model of the firm as a profit-maximizer, while compatible with the framework here, is

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With this notation, the set of all contracts in an arbitrary subset $Y \subseteq X$ associated with some talent $i \in I$ by $Y(i)$ (the map is $Y : I \rightrightarrows Y$) is defined by $Y(i) \equiv \{y \in Y \subseteq X : i \in I(y)\}$. Another typical practice in this paper will be the identification of singleton sets with the element it contains, as above. For any map f from domain E to codomain F, the following extension of this map over the domain 2^E will also be denoted by f: $f(E') \equiv \bigcup_{e\in E'} f(e)$ for every $E' \subseteq E$ (note that $f(\emptyset) \equiv \emptyset$). Given a subset of contracts $Y \subseteq X$, $I(Y)$ is the subset of talents associated with at least one contract in Y. Consider the following more complex example: suppose we have two subsets of contracts Y and Z, and we want to work with the set of all contracts in Z that name some talent that is named by some contract in Y. This is exactly $Z(I(Y))$, since $I(Y)$ is the set of talent that have a contract in Y, and $Z(I')$ is the set of all contracts that name a talent in the set I'.

[9]A strict preference relation on a set is complete, asymmetric, transitive binary relation on that set. A weak preference relation is a complete, reflexive, transitive binary relation.

[10]The assumption that the only determinant of choice behavior is the preference relation is in keeping with the spirit of cooperative game theory and matching theory.

not assumed.

2.2 Internal Assignments, Governance and Stability

Fix an institution k and take as given $X(k)$ and $\{X(d)\}_{d \in D(k)}$. Let $Y \subseteq X(k)$ be a choice situation for the institution k. The governance structure ψ^k determines the institution's choice from Y, $C^k(Y)$, via an **internal assignment** f_Y, which is a correspondence from $D(k)$ to Y such that the feasibility condition of one contract per talent is satisfied: $|\bigcup_{d \in D(k)} f_Y(d) \cap X(i)| \leq 1$. Any contract $y \in Y$ such that $f_Y^{-1}(y) = \emptyset$ is considered to be *unassigned* at Y. A contract $y \in Y$ may contain terms that disallow certain divisions from accessing this contract. For example, divisions may be geographical offices of a firm and the contract may specify geographical restrictions. Any such restrictions are respected by ψ^k and are formally captured by excluding the contract from the domain of interest of the disallowed divisions. Thus, any internal assignment f_Y will respect these contract restrictions. Let F_Y be the set of all internal assignments given $Y \subseteq X(k)$ and let $F \equiv \bigcup_{Y \subseteq X(k)} F_Y$ be the set of all internal assignments. The institutional choice from Y given some internal assignment f_Y is defined as $C^k(Y; f_Y) \equiv \bigcup_{d \in D(k)} f_Y(d)$. Note that given Y, all unassigned contracts are *rejected* from Y.

Given a choice situation Y and the list of preferences of divisions $\mathcal{P}(k)$, the governance structure ψ^k determines an **internally stable** assignment $\psi^k(Y, \mathcal{P}^d) \in F_Y$.[11] For this paper I focus on governance structures that satisfy **institutional efficiency** i.e. for any Y, if f_Y is internally stable, then there does not exist $f'_Y \in F_Y$ such that $f'_Y \mathcal{R}^d f_Y$ for all $d \in D(k)$ and $f'_Y \mathcal{P}^d f_Y$ for some d. Let Ψ^k be the family of institutionally efficient governance structures for k.

2.3 Market Outcomes, Governance and Stability

For the sake of notational convenience, I extend the definition of choice functions for talents and institutions to choice situations where contracts not naming them are present: for any $Y \subseteq X$ and for any $j \in I \cup K$, $C^j(Y) \equiv C^j(Y(j))$. So, for a choice situation the only contracts that matter for j are those contracts that name it.

A market **outcome** (or **allocation**) is a feasible collection of contracts $A \subseteq X$, i.e. for all $i \in I$, $Y(i) \in \mathcal{X}(i)$. Let $\mathcal{A}$ be the set of all feasible outcomes. I extend preferences of talent from $\mathcal{X}(i)$ to $\mathcal{A}$ (keeping the same notation for the relations) as follows: for any $i \in I$ and $A, A' \in \mathcal{A}$, $A\, P(i)\, A'$ if $A(i)\, P(i)\, A'(i)$ and $A\, R(i)\, A'$ if $A(i)\, R(i)\, A'(i)$. So, talents are indifferent about the presence or absence of contracts in an outcome that do not name them.

The market governance structure within which talents and institutions transact determines what each of these market participants are capable of securing. That a talent is free to contract with any

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[11]One could allow for multiple internally stable assignments but I focus in this paper on single-valuedness.

institution, or not at all, is an outcome of the market governance structure enabling this. Similarly, that an institution may cancel a contract with a talent also reflects the rules of the marketplace. In matching theory, and cooperative game theory more generally, this is modeled by describing the way in which a market outcome can be *blocked* or *dominated*. Thus, any market outcome that is not blocked is considered to be consonant with the rules of market governance, and is considered *stable*. An important question is whether a given market governance structure, together with the interests and behavior of the market participants, allows for stable market outcomes.

An outcome A is **individually rational** for talent i if $A(i) \, R(i) \, \emptyset$. This captures the notion that i is not compelled to participate in the market by holding a contract that he prefers less than his outside option. An outcome A is **institutionally blocked** by institution k if $C^k(A(k)) \neq A(k)$. This captures the notion that k can unilaterally sever relationships with some talent without disturbing relationships with other talents and that the outcome has to be consistent with internally stable assignments. An outcome A is **institutionally stable** if it is not institutionally blocked by any institution. An outcome A is **individually stable** if it is individually rational for all talent and institutionally stable at every institution.

An outcome A is **pairwise blocked** if there exists a contract $x \in X \backslash A$ such that the talent $I(x)$ strictly prefers outcome $A \cup \{x\}$ to A and the institution $K(x)$ will choose this contract from $A \cup \{x\}$, that is $x \in C^{I(x)}(A \cup \{x\})$ and $x \in C^{K(x)}(A \cup \{x\})$. This captures the notion that the possibility of a new mutually chosen relationship will upset an outcome, and so the initial outcome is not secure. An outcome A is **pairwise stable** if it is individually stable and it is not pairwise blocked.

An outcome A is **setwise blocked** if there exists a *blocking set* of contracts $Z \subseteq X \backslash A$ such that every talent $i \in I(Z)$ strictly prefers $A \cup Z$ to A and every institution $k \in K(Z)$ will choose all its contracts in Z from choice situation $A \cup Z$ i.e. for all $i \in I(Z)$, $Z(i) \in C^i(A \cup Z)$ and for all $k \in K(Z)$, $Z(k) \subseteq C^k(A \cup Z)$. This captures the notion that the possibility of a collection of new relationships that would be chosen if available together with existing relationships will upset an allocation. An outcome A is **stable** if it is individually stable and it is not setwise blocked.

An outcome A is **dominated by** A' **via** J, where A' is an alternate outcome and $J \subseteq I \cup K$ is a deviating coalition, if

1. the deviating coalition's contracts in the alternate outcome is different from that in the original allocation: $A'(J) \neq A(J)$.
2. every deviating actor $j \in J$ holds contracts with other deviating actors only: for all $i \in J \cap I$, $K(A'(i)) \in J$, and for all $k \in J \cap K$, $I(A'(k)) \subseteq J$.
3. every deviating actor $j \in J$ would choose its contracts in the alternate outcome A' over those in the original outcome: for all $i \in J \cap I$, $C^i(A \cup A') = A'$ and for all $k \in J \cap K$, $C^k(A \cup A') = A'$.

An outcome A is in the **core** (is **core stable**) if there does not exist another outcome that dominates it via some coalition.

The concept of pairwise stability was first introduced by Gale and Shapley (1962), in a setting where pairwise stability and (setwise) stability coincide. Like the cooperative game concept of the core, the solution concept of stability appeals to outcomes of the economy to generate predictions, without considering strategic aspects that require the level of detail common in non-cooperative game theory. The stability concepts are closer in spirit to the concept of competitive equilibrium; in the stability concept the choice situation is taken as given just as in the competitive equilibrium concept the prices are taken as given (see Ostrovsky (2008) for an elaboration of this argument in the context of supply chain markets).

In the present setting of many-to-one matching, the set of core outcome and the set of stable outcome coincides. This is the content of the following lemma, analogues[12] of which have been proved in many-to-one matching settings where choice is generated by preferences for all market participants.

Lemma 1. *An outcome is in the core if and only if it is stable.*

Proof. First, we will show that every stable outcome is in the core, by proving the contrapositive. Suppose A is dominated by A' via coalition J. Suppose J contains no institution. Then, every deviating talent receives his outside option, and by domination requirement 1 at least one of these deviators held a different contract in A than the null contract $\emptyset$ in A'. Pick one such talent $i \in J$. Then A is not individually rational for i and so A is not stable. Instead, suppose J contains at least one institution k. If every institution holds exactly the same set of contracts in A' and A, then we are back to the case where at least one worker holds a different contract in A and A'. Moreover, it must be the case, given all $k \in J \cap K$ hold the same contracts in A and A', that this one worker holds the null contract in A', and so again we have that A is not individually rational for this worker and hence not stable. So, in the final case, we have at least one institution $k \in J$ and moreover this institution holds different contracts in A and A'. Then the set of contracts $Z \equiv A'(k)$ constitutes a block of A, since domination condition 3 implies $C^k(A \cup Z) = Z$ and $C^i(A \cup Z) = Z$ for any $i \in I(Z)$, proving A is not stable.

Second, we will show that every core outcome is stable, by proving the contrapositive. Suppose A is setwise blocked by $Z \subseteq X \backslash A$. Define $J \equiv I(Z) \cup K(Z)$ and for each $j \in J$, define $B_j \equiv C^j(A \cup Z)$. Define $A' \equiv \left(A \backslash \bigcup_{j \in J} A(j) \right) \cup \left(\bigcup_{j \in J} B_j \right)$. Note that A' is an outcome by construction. Now, define $J' \equiv I \left(\bigcup_{k \in K \cap J} (B_j \backslash Z) \right)$; these are the talents not in the blocking coalition J whose contracts with blocking institutions are chosen after the block. There is no analogous

[12]See Echenique and Oviedo (2004) for a proof of this in the classic many-to-one matching model, and see Hatfield and Milgrom (2005) for a similar statement.

set of institutions, since the unit-demand condition of talents' preferences implies that blocking talents do not hold any contracts with non-blocking institutions after the block. It follows from the construction of A' that A is dominated by A' via coalition $J \cup J'$. □

This coincidence of the more widely-known concept of the core with the matching solution concept of stability supports the argument that stability is an important condition for market outcomes to satisfy. In the Walrasian model of markets, similar results relating the core to the competitive equilibrium lend support to the latter as a market outcome. While in that setting equivalence of the two does not hold generally, the core convergence result of Debreu and Scarf (1963) shows that in sufficiently large markets every core outcome can be supported as a competitive equilibrium outcome and vice versa, and provides a proof of the Edgeworth conjecture. Similar large market results have been obtained in matching models.[13]

2.4 Conditions on Preferences and Choice

Certain conditions on choice are needed to ensure existence of stable outcomes in many-to-one matching models.[14] Perhaps the most important of these conditions is substitutability.

Definition 1 (Substitutability). *A choice function C^k on domain $X(k)$ satisfies* ***substitutability*** *if for any $z, x \in X(k)$ and $Y \subseteq X(k)$, $z \notin C^k(Y \cup \{z\})$ implies $z \notin C^k(Y \cup \{z, x\})$.*

Substitutability, introduced in its earliest form by Kelso and Crawford (1982), is sufficient for the existence of stable outcomes in many-to-one matching models when choice is determined by preferences, both in the classical models without contracts and in the more general framework with contracts, this last result due to Hatfield and Milgrom (2005). In addition, the set of stable matchings has a lattice structure, with two extremal stable matchings, each distinguished by simultaneously being the most preferred stable matching of one side and the least preferred stable matching of the other side.

Substitutability has also proved useful as a sufficient condition for existence of weakly setwise stable outcomes in the many-to-many matching with contracts model, a concept introduced and studied in Klaus and Walzl (2009). These authors follow the early literature in assuming that contracts are comprehensive, so that any pair has at most one contract with each other in an outcome. Hatfield and Kominers (2012a) instead assume that a pair may have multiple contracts with each

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[13]See Kojima and Pathak (2009) and Azevedo and Leshno (2012).

[14]For the sake of collecting definitions in one subsection, I define and discuss the important conditions on choice that will be used in this paper. The reader may wish to skip these and proceed to the next section on hierarchical institutions, using this subsection as a useful reference.

other in an outcome and show that substitutability is sufficient under their definition of stability.[15] Substitutability is not sufficient for existence of outcomes that satisfy a solution concept stronger than weak setwise stability, though Echenique and Oviedo (2006) show that strengthening the condition for one side to strong substitutes restores existence for this stability notion in the classical setting.

While providing the maximal Cartesian domain for existence of stable outcomes in the classical many-to-one matching model (the college admissions model), substitutability is not the weakest condition ensuring existence of stable outcomes in many-to-one matching with contracts. Hatfield and Kojima (2010) provide a weaker substitutability condition that ensures existence of stable outcomes in models with preferences as primitives.

Definition 2 (Bilateral Substitutability). *A choice function C^k on domain $X(k)$ satisfies **bilateral substitutability** if for any $z, x \in X(k)$ and $Y \subseteq X(k)$ with $I(z) \notin I(Y)$ and $I(x) \notin I(Y)$, $z \notin C^k(Y \cup \{z\})$ implies $z \notin C^k(Y \cup \{z, x\})$.*

Bilateral substitutability guarantees existence in the many-to-one setting, but the structure of the stable set is no longer a lattice, and extremal outcomes need not exist. Hatfield and Kojima (2010) provide an intermediate condition, unilateral substitutability, that restores the existence of one of the extremal stable outcome, the doctor-optimal stable outcome, which is simultaneously the hospital-pessimal stable outcome.[16]

Definition 3 (Unilateral Substitutability). *A choice function C^k on domain $X(k)$ satisfies **unilateral substitutability** if for any $z, x \in X(k)$ and $Y \subseteq X(k)$ with $I(z) \notin I(Y)$, $z \notin C^k(Y \cup \{z\})$ implies $z \notin C^k(Y \cup \{z, x\})$.*

Bilateral substitutability does not provide a maximal Cartesian domain for sufficiency of existence, unlike substitutability in the college admissions model. Hatfield and Kojima (2008) introduced the weak substitutes condition, which mimics substitutability for a *unitary set* of contracts, defined to be a set in which no talent has more than one contract present. The authors show that any Cartesian domain of preferences that guarantees existence of stable outcomes must satisfy weak substitutability.

Definition 4 (Weak Substitutability). *A choice function C^k on domain $X(k)$ satisfies **weak substitutability** if for any $z, x \in X(k)$ and $Y \subseteq X(k)$ with $I(z) \notin I(Y)$, $I(x) \notin I(Y)$ and $|I(Y)| = |Y|$, $z \notin C^k(Y \cup \{z\})$ implies $z \notin C^k(Y \cup \{z, x\})$.*

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[15]The stability definition of Hatfield and Kominers (2012a) coincides with the weak setwise stability of Klaus and Walzl (2009) under the assumption of comprehensive contracts (which Kominers (2012) call *unitarity*), but is stronger under the assumption of non-comprehensive contracts. They also prove that substitutability provides a maximal Cartesian domain for existence of stable outcomes, with the caveat that contracts are not comprehensive.

[16]In their setting, doctors are the talents who can hold only one contract in an outcome and hospitals are the institutions which can hold many contracts in an outcome. Moreover, hospitals have preferences as primitives that define choice behavior.

The common assumption about choice behavior in the matching literature has been that agents choose by maximizing a preference relation or objects are allocated while respecting a priority relation. With the definition of stability introduced in Hatfield and Milgrom (2005), however, one that makes reference only to choice functions, it is no longer necessary to make reference to underlying preferences for the model to be studied, since substitutability is a condition on choice functions as well. For this more abstract setting however, substitutability is no longer a sufficient condition for existence, as shown by Aygün and Sönmez (2012b). These authors introduce the Irrelevance of Rejected Contracts condition on choice that restores the familiar results of matching models under substitutable preferences, such as the lattice structure and the opposition of interests at extremal matchings.

Definition 5 (Irrelevance of Rejected Contracts). *A choice function C^k on domain $X(k)$ satisfies the **Irrelevance of Rejected Contracts (IRC)** condition if for any $Y \subseteq X(k)$ and $z \in X(k) \backslash Y$, $z \notin C^k(Y \cup \{z\})$ implies $C^k(Y \cup \{z\}) = C^k(Y)$.*

Choice derived from preferences must satisfy the Strong Axiom of Revealed Preference (SARP)[17], and it is the combination of this choice assumption and substitutability that yields the results of Hatfield and Milgrom (2005). However, under the substitutes condition, IRC is no weaker than SARP. However, the IRC condition is also sufficient to restore all the results of Hatfield and Kojima (2010) under the weaker substitutes conditions introduced therein, and Aygün and Sönmez (2012a) also show that in this setting IRC is strictly weaker than SARP.

While substitutability and unilateral substitutability are strong enough conditions to provide useful structure on the stable set, particularly in ensuring the existence of a talent-optimal stable outcome, they are not strong enough to yield the result that a strategyproof mechanism exists for this domain, a result that is familiar from the college admissions model with responsive preferences. Hatfield and Milgrom (2005) show that under a condition on choice they call Law of Aggregate Demand, a generalized version of the Gale-Shapley Deferred Acceptance algorithm serves as a strategyproof mechanism for talent.

Definition 6 (Law of Aggregate Demand). *A choice function C^k on domain $X(k)$ satisfies the **law of aggregate demand (LAD)** if for any $Y, Y' \subseteq X(k)$, $Y \subseteq Y'$ implies $|C^k(Y)| \leq |C^k(Y')|$.*

Alkan (2002) introduced the analog of this condition, *cardinal monotonicity*, for the classical matching model to prove a version of the *rural hospital theorem*[18]. He demonstrates that with cardinal monotonicity, in every stable matching every agent is matched to the same number of partners. The

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[17] In a matching setting, where choice is combinatorial, a choice function C with domain X satisfies the **Strong Axiom of Revealed Preference** (**SARP**) if there does not exist a sequence of distinct $X_1, \ldots, X_n, X_{n+1} = X_1$, $X_m \subseteq X$, with $Y_m \equiv C(X_m)$ and $Y_m \subseteq X_m \cap X_{m+1}$ for all $m \in 1, \ldots, n$.

[18]Roth (1986) showed that in the college admissions model with responsive preferences, any college that does not fill its capacity in some stable matching then in every stable matching it is matched to exactly the same set of students.

analog for the contracts setting is that under the Law of Aggregate Demand, every institution holds the same number of contracts in every stable outcome.

One last condition that will prove useful in the later section on a decentralized bargaining game is the condition of Pareto Separable choice.

Definition 7 (Pareto Separable). *A choice function C of an institution k (or division d) is Pareto Separable if, for any $i \in I$ and distinct $x, x' \in X(i, k)$, $x \in C(Y \cup \{x, x'\})$ for some $Y \subseteq X(k)$ implies that $x' \notin C(Y' \cup \{x, x'\})$ for any $Y' \subseteq X(k)$.*

Hatfield and Kojima (2010) prove that substitutability is equivalent to unilateral substitutability and the Pareto Separable condition. A partial analog to this result is that weak substitutability and the Pareto Separable condition is equivalent to bilateral substitutability, though the converse is not true.

Proposition 1. *Suppose institution k has a choice function C satisfying IRC, weak substitutes and the Pareto Separable condition. Then C satisfies bilateral substitutes.*

The Pareto Separable condition states that if in a choice situation some contract with a talent is not chosen but an alternative contract with this talent is, then in any other choice situation where the alternative is present the first cannot be chosen. So, in particular, suppose a new contract with a new talent becomes available and is chosen. With the Pareto Separable assumption, we can conclude that there cannot be any *renegotiation with held talents*, since such a renegotiation would involve a violation of this assumption. Therefore, given the assumption of IRC, we can remove these unchosen alternatives with talents held in the original choice situation without altering choice behavior. Moreover, IRC allows us to remove any contracts with talents who are not chosen in either the original situation or in the new situation with the arrival of a previously unseen talent. Thus, we can reduce the set of available contracts in the original situation to contain no more than one contract per talent. Thus, if any previously rejected talent (or contract) is recalled with the arrival of a new talent (violating bilateral substitutes), then this behavior would prevail in the pruned choice situation, resulting in a violation of weak substitutes. This argument is formalized in the following proof.

Proof. Let $Y \subseteq X(k)$ and $z, x \in X(k) \backslash Y$ such that $z \neq x$ and $I(z) \neq I(x)$. Moreover, suppose $I(z), I(x) \notin I(Y)$. Suppose $z \notin C(Y \cup \{z\})$. Now, suppose $z \in C(Y \cup \{z, x\})$, which constitutes a violation of bilateral substitutability. First, suppose there exist $w \in Y$ such that $w \notin C(Y \cup \{z\})$ and $w \notin C(Y \cup \{z, x\})$. Then by IRC we can remove w from Y without affecting choice i.e. $C(Y' \cup \{z\}) = C(Y \cup \{z\})$ and $C(Y' \cup \{z, x\}) = C(Y' \cup \{z, x\})$. Repeatedly delete such contracts, and let Y' denote the set remaining after all such deletions from Y.

If there exist $y, y' \in Y'$ with $I(y) = I(y')$ such that $y \in C(Y' \cup \{z\})$ and $y' \in C(Y' \cup \{z, x\})$, then C would violate the Pareto Separable condition, given that no more than one contract with $I(y)$ can be chosen. Thus, if $y \in C(Y' \cup \{z\})$ then for any $y' \in Y'$ with $I(y') = I(y)$, $y' \notin C(Y' \cup \{z, x\})$. So, by IRC, $C(Y'' \cup \{z\}) = C(Y' \cup \{z\})$ and $C(Y'' \cup \{z, x\}) = C(Y' \cup \{z, x\})$, where $Y'' = Y' \backslash \{y'\}$. We can repeat this deletion procedure and let Y'' denote the set remaining after all such deletions from Y.

It should be clear that $|Y''| = |I(Y'')|$. Moreover, we have that $z \notin C(Y'' \cup \{z\})$ but $z \in C(Y'' \cup \{z, x\})$, constituting a violation of weak substitutes, and concluding our proof.

$\square$

3   The Theory of Hierarchical Institutions

In this section, I define and examine a particular institutional governance structure, the *inclusive hierarchical* governance structure. Unlike the market governance structure, which is a rather permissive type of governance structure that allows talents and institutions to freely recontract, inclusive hierarchical governance structures greatly enhance the bargaining power of divisions versus talents. The view taken in this section is that talents are human resources to be allocated within the institution, and the institutional governance structures considered reflects this aim. The inclusive hierarchical governance structure provides talents with *weak veto power* since they can leave any contract with the institution for another institution, is *institutionally efficient* since there does not exist any internal assignment of contracts to divisions that is weakly improving for every division and strictly improving for some, and is *situationally strategyproof* since for a fixed take-it-or-leave-it choice situation every division has a dominant strategy reveal its preferences when the governance structure ψ is viewed as a mechanism. Proofs of results can be found in the appendix.

3.1   The Inclusive Hierarchical Governance Structure

A governance structure $\psi \in \Psi^k$ has a **hierarchy** if it is parametrized by a linear order $\rhd^k$ on $D(k)$. **Inclusive Hierarchical (IH)** governance structures constitute a class of governance structures where the hierarchy $\rhd^k$ determines how conflicts between divisions over contracts are resolved, and where divisions have the power to choose contracts without approval of other divisions, except in the case of conflicts for talents already mentioned. For example, given a choice situation Y, if there is a contract $y \in Y$ such that distinct divisions $d, d' \in D(k)$ both have y as part of their most preferred bundle of contracts in Y, then the governance structure resolves this conflict in favor of the division with higher rank, where $d \rhd^k d'$ means that division d has a higher rank than d'. However, if given any two divisions their most preferred bundles in Y are such that there is no

conflict over a contracts or talents, then the divisions have the autonomy to choose these bundles on behalf of the institution. The order $\rhd^k$ defines a *ranking* of divisions, where division d is said to be *higher-ranked* than division d' if $d \rhd^k d'$, where $d, d' \in D(k)$ for some institution k. Since it should not cause any confusion, let $\rhd^k : D(k) \to \{1, \ldots, |D(k)|\}$ be the rank function, where $\rhd^k(d) < \rhd^k(d')$ if and only if $d \rhd^k d'$. Also, for any $n \in \{1, \ldots, |D(k)|\}$, let d_n^k denote the n-th ranked division i.e. $\rhd^k(d_n^k) = n$.[19]

The inclusive hierarchical governance structure ψ^k parametrized by $\rhd^k$ can be modeled using the following choice aggregation procedure, the **inclusionary hierarchical procedure**. This procedure determines the internal assignment of contracts for a given choice situation $Y \subseteq X(k)$, and thence the derived institutional choice $C^k(Y)$. The procedure is analogous to a serial dictatorship in the resource allocation literature, with the hierarchy $\rhd^k$ serving as the serial ordering. The highest ranked division d_1^k is assigned its most preferred set of contracts from Y. The next highest ranked division d_2^k is assigned its most preferred set of contracts from the remain set of contracts, and so on. Importantly, after a division's assignment is determined, any unassigned contracts that name a talent assigned at this step are removed (though still unassigned), and the remaining contracts constitute the availability set for the next step. At every step, the assignment must be feasible, so that no division d is assigned a contract outside of its domain of interest $X(d)$.

The formal description of the procedure requires some notation. Let $Y \subseteq X(k)$ be a subset of contracts naming the institution k. There are $N^k = |D(k)|$ steps in the procedure. For the sake of notational convenience and readability, I will suppress dependence on the institution k, which will be fixed. For any $n \in \{1, \ldots, N\}$, let λ_n^Y be the set of contracts *available at step n*, let α_n^Y be the set of contracts *available and allowed at step n*, β_n^Y be the set of contracts *available and not allowed at step n*, γ_n^Y be the set of contracts *assigned at step n*, δ_n^Y be the set of contracts *eliminated at step n*, and ρ_n^Y be the set of contracts *rejected at step n*.

Step 1 Define $\lambda_1^Y \equiv Y$. Define $\alpha_1^Y \equiv \lambda_1^Y \cap X(d_1)$, $\beta_1^Y \equiv \lambda_1^Y \backslash \alpha_1^Y$, $\gamma_1^Y \equiv C^{d_1}(\alpha_1^Y)$, $\delta_1^Y \equiv \left(\lambda_1^Y \cap X(I(\gamma_1^Y)) \right) \backslash \gamma_1^Y$, and $\rho_1^Y \equiv \alpha_1^Y \backslash (\gamma_1^Y \cup \delta_1^Y)$.

$$\vdots$$

Step n Define $\lambda_n^Y \equiv (\beta_{n-1}^Y \backslash \delta_{n-1}^Y) \cup \rho_{n-1}^Y$. Define $\alpha_n^Y \equiv \lambda_n^Y \cap X(d_n)$, $\beta_n^Y \equiv \lambda_n^Y \backslash \alpha_n^Y$, $\gamma_n^Y \equiv C^{d_n}(\alpha_n^Y)$, $\delta_n^Y \equiv \left(\lambda_n^Y \cap X(I(\gamma_n^Y)) \right) \backslash \gamma_n^Y$, and $\rho_n^Y \equiv \alpha_n^Y \backslash (\gamma_n^Y \cup \delta_n^Y)$.

The *internal assignment* $f_Y(d)$ of division $d \in D(k)$ given a choice situation Y is $f_Y(d) = \gamma_{\rhd^k(d)}^Y$. The derived *institutional choice* $C^k(Y)$ from set Y is defined by $C^k(Y) \equiv \bigcup_{n=1}^{N^k} \gamma_n^Y$. Note that both f_Y and $C^k(Y)$ depend upon the hierarchy $\rhd^k$.

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[19]A division d higher-ranked than another division d' if and only if its rank number $\rhd^k(d)$ is *smaller* $\rhd^k(d')$.
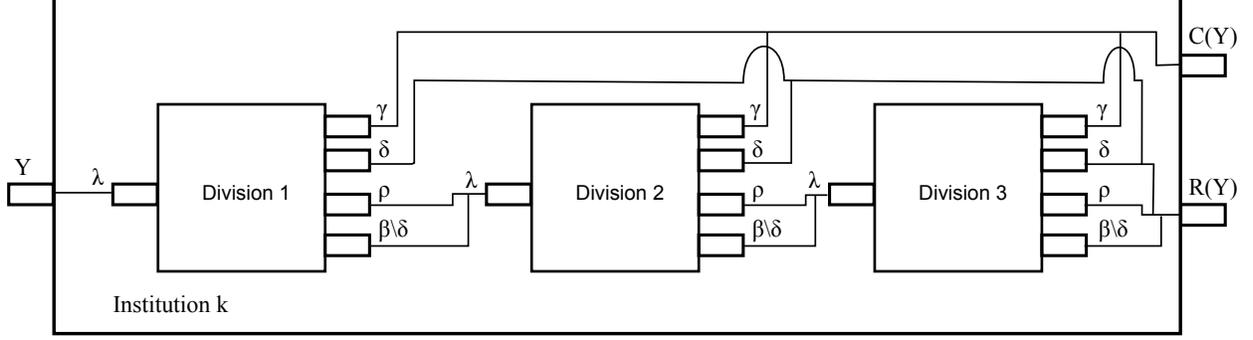
Figure 1: Graphical Depiction of a Hierarchical Institution with three Divisions, with the various contract-pathways of the Inclusionary Hierarchical Procedure displayed.

Figure 1 illustrates the inclusionary hierarchical procedure for an institution with three divisions. In this case, the choice procedure has three steps, one for each division. One can imagine that the set of contracts available to the institution "flow" through the institution along the "paths" illustrated, where divisions "split" the flow into various components that then travel along different paths. Some of these paths meet at a "union junction" (every junction in this figure is a union junction); some paths lead to a division of the institution. The paths form an "acyclic network" beginning at the "entry port" of the institution and ending at either the "acceptance port" or "rejection port", and so every contract that enters the institution will exit after encountering a finite number of nodes. While this description choice is not meant to be taken literally, it is a useful mnemonic for understanding the forthcoming results.

In summary, for any choice situation $Y \subseteq X(k)$, the internal assignment f that is internally stable given an inclusive hierarchical governance structure ψ^k with hierarchy $\rhd^k$ coincides with the assignment $\left(\gamma_n^Y\right)_{n=1}^{N^k}$ produced by the corresponding inclusionary hierarchical procedure.

3.2 Properties of Inclusive Hierarchical Governance

I now turn to answering the main question posed by this paper: why hierarchies? In this subsection I will demonstrate that inclusive hierarchical governance structures have the positive property that the institutional choice function derived from the internally stable assignment satisfies two key choice properties, the Irrelevance of Rejected Contracts and bilateral substitutability, under the assumption that divisions have bilaterally substitutable preferences. This important result will then straightforwardly lead to the theorem that markets featuring institutions with inclusive hierarchical governance are guaranteed to have stable outcomes. Other interesting results about this governance structure will also be discussed.

Fix an institution k with divisions $D(k)$, where $(\mathcal{P}^d)_{d \in D(k)}$ are the preferences of each divi-

sion, which respect the domain of interest restrictions $\mathcal{D}(k)$.[20] Let ψ^k be the inclusive hierarchical governance structure of k, parameterized by $\rhd^k$. In order to ease exposition and readability, I will suppress notation indicating the institution. Thus, for the purposes of this subsection, we will denote $X(k)$, the set of all contracts naming institution k, simply by X, and $D(k)$, the set of all divisions in k, simply by D.

The first property of inclusive hierarchical choice aggregation is that the IRC property of division choice will be preserved at the institutional level. As discussed previously, this condition states that the presence of "dominated" contracts in particular choice situation has no bearing on the choice, and so their removal from the available set does not alter the chosen set.

Theorem 1. *The institutional choice function C derived from the inclusive hierarchical governance structure parametrized by $\rhd^k$ satisfies the IRC condition if for every division $d \in D$, C^d satisfies the IRC condition.*

The following theorem is the key choice property with inclusive hierarchical governance. The property of bilateral substitutes is preserved by aggregation, given that divisional choice satisfies it and IRC.

Theorem 2. *The institutional choice function C derived from the inclusive hierarchical governance structure parametrized by $\rhd^k$ satisfies bilateral substitutes if for every division $d \in D$, C^d satisfies bilateral substitutes and the IRC condition.*

The important observation in the proof is that an expansion of the choice situation through the introduction of a contract with a new or unchosen talent improves the array of contract options for every division in the institution, and not just for the highest-ranked division, given the assumptions of bilateral substitutability and IRC of division choice.

It is also the case that choice aggregation with inclusive hierarchies preserves the property of weak substitutes.

Proposition 2. *The institutional choice function C derived from the inclusive hierarchical governance structure parametrized by $\rhd^k$ satisfies weak substitutes if for every division $d \in D$, C^d satisfies weak substitutes and the IRC condition.*

The proof follows a similar strategy to that of Theorem 2, showing a monotonic relationship between certain choice situations of the institution and the resultant choice situations of each division.

Intriguingly, this preservation by inclusive hierarchical aggregation does not hold when divisions have substitutable choice, as shown by Kominers and Sönmez (2012) in the slot-specific priorities

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[20]The results of this subsection also hold if division choice is taken to be primitive with the additional assumption of IRC.

model, where slots are analogous to unit-demand divisions and the order of precedence is analogous to the institutional hierarchy. They provide an example where institutional choice violates substitutes and unilateral substitutes with two divisions of unit-demand. These authors also obtain results that correspond to Theorems 1 and 2 and Proposition 2. It is also the case that the unilateral substitutes property cannot be preserved through this aggregation. Thus, bilateral substitutes is the strongest substitutability property that is preserved through inclusive hierarchical governance.

That the property of weak substitutes is preserved through aggregation leads naturally to the following result for the classical matching setting, since weak substitutes is a property that places conditions on choice in situations where no talent has more than one contract available.

Proposition 3. *If $X(k)$ is a classical contract set and if C^d satisfies Subs and IRC for all $d \in D(k)$, then C^k satisfies Subs and IRC.*

Another novel result of the inclusive hierarchical procedure is that SARP is preserved. Thus, in the baseline case where divisions are assumed to have preferences, the institutional choice can in fact be rationalized by some preference relation. Nevertheless, as shown in Aygün and Sönmez (2012a), there exist unilaterally substitutable choice functions that satisfy IRC and the law of aggregate demand that violate the SARP, and so if divisional choice was not generated by preferences it could well be that the institutional choice cannot be rationalized either.

Theorem 3. *The institutional choice function C derived from the inclusive hierarchical governance structure parametrized by $\rhd^k$ satisfies SARP if for every division $d \in D$, C^d satisfies SARP.*

The following is an example of a bilaterally substitutable and IRC choice function that cannot be decomposed into a sequential dictatorship of unit-demand divisions with strict preference relations. In fact, it cannot be non-trivially generated by an institution with at least two divisions with bilaterally substitutable choice functions.

Example 1. Suppose we have a choice function C defined as follows:

$$C(\emptyset) = \emptyset$$

$C(x) = x$	$C(x') = x'$	$C(z) = z$	$C(z') = z'$
$C(\{x, x'\}) = x$	$C(\{x, z\}) = x$	$C(\{x, z'\}) = x$	
$C(\{x', z\}) = z$	$C(\{x', z'\}) = \{x', z'\}$	$C(\{z, z'\}) = z$	
$C(\{x, x', z\}) = x$	$C(\{x, x', z'\}) = \{x', z'\}$	$C(\{x, z, z'\}) = x$	$C(\{x', z, z'\}) = \{x', z'\}$
	$C(\{x, x', z, z'\}) = \{x', z'\}$		

Contracts x and x' are with talent t_x and contracts z and z' are with talent t_z.

Since x' and z' are selected from the largest offer set, one of these two contracts must be the highest priority (amongst contracts with these two talents) for the division with the highest rank that

ever holds a contract with any one of these two talents. Without loss of generality, suppose it is x'. Then, since x' will always be picked by this division over any contract with talents t_x, t_z, if available, it must be that contract x' is never rejected. But this is not the case for choice function C, proving that this choice function cannot be generated by a sequential dictatorship of unit-demand divisions.

The key feature of this example is that $\{x', z'\}$ are complementary. This is illustrated by supposing there are two divisions d and d', where $d \rhd d'$, with preferences $\{x', z'\} \succ_d \emptyset$ and $x \succ_{d'} z' \succ_{d'} z \succ_{d'} x' \succ_{d'} \emptyset$; the institutional choice function is identical to C. However, in this case, the choice function of the first division does not satisfy bilateral substitutes (in fact, violates weak substitutes). Furthermore, there does not exist any *non-trivial* institution with at least two divisions that generates this choice function. Thus, we have shown that there exist bilaterally substitutable choice functions that cannot be generated from a non-trivial inclusive hierarchy with bilaterally substitutable divisions.

Proposition 4. *In the setting with classical contracts, if C^d satisfies substitutability and the LAD for every $d \in D$ and the set of acceptable talents $X(d)$ is the same for every division, then with inclusive hierarchical governance the derived choice function C satisfies substitutability and LAD.*

Proof. Let $Y \subseteq X$ and $z \in X \backslash Y$. Define $Z \equiv Y \cup \{z\}$. The first thing to note is that C_d satisfies IRC since it satisfies Subs and LAD. Thus, from Proposition 1, C satisfies IRC. Thus, if $x \notin C(Z)$, then $C(Z) = C(Y)$ and so the condition for LAD is satisfied. So, suppose $z \in C(Z)$. Now, consider the first division according to $\rhd$. If z is rejected, then the division chooses exactly the same contracts it would choose with z present, and so the cardinality of the set of contracts rejected by the division increases by exactly one, and the cardinality of the chosen set stays the same. If z is accepted, then by the Subs condition, every previously rejected contract remains rejected and by LAD the cardinality of the chosen set does not decrease. Thus, these restrictions imply that at most one previously chosen contract is now rejected due to the acceptance of z, and so the set of contracts rejected by the first division increases by at most one contract. Next, suppose that the set of contracts rejected increases by at most one for every division up to and including k. Then, the previous argument can be repeated to show that the set of rejected contracts increases by at most one, thereby demonstrating that that set of contracts which are unchosen using C increases in cardinality by at most one, and so we have that C satisfies LAD. $\square$

Corollary 1. *In the setting with classical contracts, if every division $d \in D$ has unit-demand with strict preferences and the set of acceptable talents is the same for every division, then C satisfies Subs and LAD.*

Proof. This follows from the observation that the condition of unit-demand with strict preferences induces a substitutable choice function for the division satisfying LAD, combined with the previous proposition. $\square$

3.3 On Markets and Hierarchies

With the results of the previous subsection, we know that an institution with an inclusive hierarchy will have a derived choice function that satisfies the properties of IRC and bilateral substitutability, amongst other properties. Consider now an economy with some set of institutions K, each of which is organized by an inclusive hierarchy of divisions, and some set of talents I and some set of contracts X. The key existence result for this economy is that the set of stable market outcomes, and so the core, is nonempty.

Theorem 4. *If for every institution $k \in K$ the choice functions C^d of every division $d \in D(k)$ satisfies IRC and bilateral substitutability, then the set of stable market outcomes is nonempty.*

Proof. By Theorem 2, we know that choice function of every institution satisfies IRC and bilateral substitutability. Then by Theorem 1 of Hatfield and Kojima (2010), the conditions of which are satisfied by the talent-institution matching economy, the set of stable outcomes is nonempty. □

The existence of a market stable outcome means that there does not exist any group of talents and divisions that can find an arrangement each of them prefers that is institutionally stable. It may be the case that some talent and division wish to hold a contract with each other, but this does not block the market outcome because the institution to which the division belongs prevents such a block from being secure. As we shall see in the next subsection, it is a property of inclusive hierarchical governance that a market stable outcome exists, and not merely that there is an institutional governance structure, even though the presence of a governance structure can limit the types of blocks to market outcomes that might be possible.

3.4 Non-Hierarchical Conflict Resolution

With inclusive hierarchical governance in institution k, conflicts between divisions over contracts are resolved through hierarchical ranking $\rhd^k$, with division d obtaining a favorable resolution in any dispute with division d' if and only if $d \rhd^k d'$. In this subsection, I will consider a more flexible conflict resolution system, where conflicts over a particular contract are resolved in a manner that is dependent on the contract in question.

Fix an institution k and now suppose that there exists a collection $\left(\rhd^k_x\right)_{x \in X(k)}$ of linear order on $D(k)$. The role of any order $\rhd^k_x$ in the institutional governance is to determine which division can claim contract x in a conflict between two or more divisions. Given some choice situation $Y \subseteq X(k)$ and contract $x \in Y$, if for some distinct $d, d' \in D(k)$ with $d \rhd^k_x d'$, $x \in X(d) \cap X(d')$, and if $x \in C^d(Y) \cap C^{d'}(Y)$, then the divisions are in conflict over x. This conflict is resolved in favor of the division with the higher rank according to $\rhd^k_x$, which in this case is d, which means

that an internal assignment f where x is assigned to d', $x \in f(d')$, and d would choose x given its assignment i.e. $x \in C^d(f(d) \cup \{x\})$ is a disputed assignment and so not internally stable.

Let ψ^k be an internally efficient governance structure parametrized by a flexible conflict resolution system $\left(\rhd^k_x \right)_{x \in X(k)}$. The requirement of internal efficiency, which is the condition that in any choice situation $Y \subseteq X(k)$ there is no feasible internal allocation g such that $g(d) \mathcal{R}^d f(d)$ for all divisions $d \in D(k)$ and $g(d) \mathcal{P}^d f(d)$ for some division $d \in D(k)$, where $f \equiv \psi^k(Y, (\mathcal{P}^d)_{d \in D(k)})$.

Theorem 5. *Suppose the contracts is classical. If all divisions are unit-demand and the institutional governance structure ψ^k is internally efficient and has a flexible conflict resolution system, then the institutional choice satisfies IRC but can violate substitutability.*

Proof. For the institution k in question, let $Y \subseteq X(k)$ be the set of contracts available to it, and let $z \in X(k) \backslash Y$. Define $\hat{Y} \equiv Y \cup \{z\}$.

Given the hierarchical priority structure at situation Y, $\mathcal{H}(Y)$, we can use the hierarchical exchange mechanism ϕ with $\mathcal{H}(Y)$ to get an assignment of contracts to divisions μ by using the preferences of the divisions as an input to ϕ.

Some notation: I assume there is some fixed exogenous tie-breaking rule that determines the order in which cycles are removed in the situation where there are multiples cycles, so that only one cycle is removed per step, where such a rule always removes older cycles before younger ones. In particular, I use an exogenous ordering of the divisions to determine the ordering of cycles to be removed when there are multiple cycles at a step, where the cycles at a step are ordered for removal as follows. There is a queue for cycle removal. In every step, have all divisions point to their favorite available contract. Order all cycles that newly appear in this step by cycle-removal order and place it into the removal queue, where a new cycle enters the queue before another new cycle if it has a division in the cycle that is cycle-removal-smaller than every agent in the other cycle. Then, remove in this round the the cycle at the front of the queue. Update the control rights of any contracts whose previously controlling division has been assigned and removed. Go to the next step.

Note that in every step, if the queue as any cycles remaining, one cycle is removed, though it is not the case that in every step new cycles are created. However, in any step where the queue is empty at the beginning of the step, a new cycle must be created if there are any divisions remaining. Let $T(Y)$ be number of steps for all divisions to be assigned or removed.

Let $(\gamma_t(Y))_{t \in T(Y)}$ be the sequence of trading cycles realized by the mechanism when the set of available contracts is Y. Then, $C(Y) \equiv \bigcup_{t \in T(Y)} X(\gamma_t(Y))$. Also, $(\gamma_t(Y))_{t \in T(Y)}$ determines the internal allocation μ_Y.

Now, let us study what occurs when a new contract z is introduced. Since the hierarchical priority structure is contract-consistent, every contract $y \in Y$ has the same division controlling it in $\mathcal{H}(Y)$ and $\mathcal{H}(\hat{Y})$. Let d be the division that controls z at $\hat{Y}$.

To demonstrate that C satisfies IRC, we will assume that $z \notin C(\hat{Y})$ and prove that $C(\hat{Y}) = C(Y)$. Given that $z \notin C(\hat{Y})$, $z \notin \gamma_t(\hat{Y})$ for any $t \in T(\hat{Y})$. Since the only way that z is removed from the assignment procedure is by removal via a trading cycle and since a division that does not have z in its domain of interest is not allowed to point to it, we know that no division could have pointed to z at any step. Thus, in every step, contracts pointed to remains the same as it did in situation Y, and so $T(\hat{Y}) = T(Y)$ and $\gamma_t(\hat{Y}) = \gamma_t(Y)$. Thus, $C(\hat{Y}) = C(Y)$, proving IRC.

To show that substitutability can be violated, consider the following example. Suppose three divisions 1, 2, and 3 with preferences: $w\mathcal{P}^1 y\mathcal{P}^1 \emptyset$, $x\mathcal{P}^2 z\mathcal{P}^2 \emptyset$, and $x\mathcal{P}^3 w\mathcal{P}^3 \emptyset$. Suppose that the priority structure is $1 \triangleright_x 3 \triangleright_x 2$, $2 \triangleright_y 3 \triangleright_y 1$, $2 \triangleright_z 3 \triangleright_z 1$, and $1 \triangleright_w 2 \triangleright_w 3$. For this problem, with $Y \equiv \{x, y, z\}$, we have that $C(Y) = \{x, y\}$, but with $\hat{Y} \equiv Y \cup \{w\}$, we have $C(\hat{Y}) = \{w, x, z\}$. The problem here is that the introduction of a new contract can make some division worse off, because the new contract can result in the loss of access to a contract that that division used to get through trading, as a consequence of the partner to that trade leaving earlier, and the inheritor of the desired contract not being interested in trading with the division in question. $\square$

As demonstrated in the counterexample, the problem with more flexible conflict-resolution together with the goal of efficiency is that the resolution process might not be consistent in the way it treats a division in terms of its welfare. Even a three-way trading cycle can lead to this non-harmonious welfare impact of an extra contract opportunity, and possibly lead to complementarity of choice at the institutional level.

4   Take-it-or-leave-it Bargaining

Towards an understanding of the impact of strategic behavior by talents and by institutional actors, consider a multi-stage game form $\mathbb{G}$, where each talent makes a take-it-or-leave-it offer of a set of contracts to an institution in the first stage, and institutions choose contracts which to accept in the second stage, with the final outcome being determined by these institutional choices. I will focus on Subgame Perfect Nash Equilibria (SPNE).

While it is certainly the case that the take-it-or-leave-it assumption places a great deal of the bargaining power in the hands of the talents, it is also worth recognizing that this bargaining power is mitigated by the presence of talent competition in the first stage, enhanced by the possibility of making offers that have multiple acceptable contracts, and so effective bargaining power of any particular talent is endogenous. We shall see that the set of outcomes realizable in SPNE are pairwise stable when institutions have an inclusive hierarchical governance structure.

It is possible that SPNE outcomes are unstable, though pairwise stable. The equilibria of such outcomes feature a coordination failure on the part of talents and an institution, due to the complementarities that are present even in bilaterally substitutable preferences of a division. With a

strengthening of conditions on institutional choice to include the Pareto Separable condition, introduced by Hatfield and Kojima (2010), I obtain the stronger result of stability of SPNE outcomes. More generally, restrictions on division preferences that ensure equivalence between pairwise stability and stability ensure that SPNE outcomes are stable. This is the case when all divisions have substitutable preferences, even though the derived institutional choice fails substitutability.

There exists a literature on non-revelation mechanisms and hiring games like the take-it-or-leave-it game studied here. Alcalde (1996) studied the marriage problem using such a game form, and showed that the set of (pairwise) stable outcomes can be implemented in undominated Nash Equilibria. Alcalde et al. (1998) study a hiring game in the Kelso-Crawford setting with firms and workers where firms propose salaries for each worker in the first stage, and workers choose which firm to work given the proposed salaries. In this firm-offering take-it-or-leave-it game, they obtain implementability of the stable set in Subgame Perfect Nash Equilibria. Under the assumption of additive preferences, they show that in the worker-offering version of the hiring game, the worker optimal stable outcome is implementable in SPNE. Alcalde and Romero-Medina (2000) show SPNE implementability of the set of stable outcomes for the college admission model using the two-stage game form with students proposing in the first stage. In Sotomayor (2003) and Sotomayor (2004), the author provides SPNE implementation results for the pairwise stable set of the marriage model and the many-to-many matching (without contracts) model. Finally, Haeringer and Wooders (2011) study a sequential game form, where firms (which have capacity one) are proposers and workers can accept or reject offers, with acceptance being final, and show that in all SPNE the outcome is the worker optimal stable outcome.[21]

The side that moves first in the two-stage game has a material impact on the stability of the outcome of the game. Stability is a group rationality concept, and tests for the presence of groups of agents that can be made better off by a coordinated alternative action. When talents propose, a deviation by a worker cannot be coordinated in the SPNE solution concept, and so at most the talent and an institution (via a division) is involved in altering the outcome. In games where colleges or firms propose (see Alcalde and Romero-Medina (2000) and Alcalde et al. (1998), respectively), a deviation by a college or firm can involve a group of workers, since many "offers" can be change in a deviation. Thus, it is not surprising that SPNE outcomes of a college- or firm-proposing bargaining game are stable without any assumptions on preferences, but outcomes of a student- or worker-proposing game are only pairwise stable for this domain. Obtaining stability in this latter version requires a strengthening of assumptions to identify stability with pairwise stability.

The distinction between the college admissions model and the Kelso-Crawford model is also important to understand the implementation results in the literature. In the latter model, the pres-

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[21]They also show that if workers make decisions simultaneously, then the set of SPNE outcomes expands to include all stable outcomes and possibly some unstable ones as well.

ence of a salary component, or more abstractly of multiple potential contracts between a firm-worker pair, means that implementability should not be expected, given that as first movers the workers/talents can take advantage of their proposing power to "select out" less preferred stable outcomes. In my setting, given the weak assumptions on preferences, stability under SPNE cannot be assured, though pairwise stability can. However, for the stronger condition of Pareto Separable preferences, together with the Weak Substitutes and IRC conditions, stability of SPNE outcomes is assured, a novel result considering the weakened domain.

Throughout this section, assume that we have a hierarchical matching problem $\mathcal{E} \in \mathbb{E}^{\mathcal{H}}$, where divisions have preferences instead of merely choice functions. Also, assume that all divisions have bilaterally substitutable preferences. Suppose the game is one of complete information, so that the preferences of talents, contract sets, preferences of divisions, and the institutional hierarchies are common knowledge amongst the talents and divisions. The formal description of the game $\mathbb{G}(\mathcal{E})$ is as follows. There are two stages, the Offer stage (Stage 1) and the Internal Choice stage (Stage 2). The players are the set of talents I and the set of divisions $D \equiv \bigcup_{k \in K} D(k)$. In Stage 1, the Offer stage, every talent simultaneously makes one offer to one institution i.e. the action ω_i taken by a talent i is an element of $\Omega_i \equiv X(i)$. Let h_0 be the history of the game *at the end* of the Offer stage. Then, if $\omega \equiv (\omega_i)_{i \in I}$ is the action profile at the Offer stage, $h_0 \equiv (\omega)$.

In Stage 2, divisions choose amongst the contract offers to their institutions. Define $\omega_k \equiv X(k) \cap \bigcup_{i \in I} \omega_i$ to be the set of offers made to institution k. For each $k \in K$, label divisions in $D(k)$ according to the linear order $\rhd^k$, so that $d_m^k \rhd^k d_n^k$ if and only $m < n$, where $m, n \in \{1, \dots, |D(k)|\}$ and $d_m^k, d_n^k \in D(k)$. Define $\mathcal{G}^k(\omega)$ to be the *internal choice game* amongst divisions $D(k)$ of institution k given offers $\omega \in \Omega \equiv \prod_{i \in I} \Omega_i$. This internal choice game is a sequential game with $|D(k)|$ rounds from 1 to $|D(k)|$, where the player at round n is $d_n^k \in D(k)$ and takes action λ_n^k. Let $h_1^k \equiv h_0$ be the history at the start of the internal choice game and let h_n^k be the history of play at the start of round n, where $h_m^k \equiv (h_{m-1}^k, \lambda_{m-1}^k)$. The action that a division takes is to choose a subset of contracts from the available set of contracts at round n. Define $\Lambda_1^k(h_1^k) \equiv \omega_k$ and

$$\Lambda_{n+1}^k(h_{n+1}^k) = \Lambda_{n+1}^k((h_n^k, \lambda_n^k)) \equiv \Lambda_n^k(h_n^k) \setminus \left(\bigcup_{i' \in I(\lambda_n^k)} X(i') \right),$$

where $\Lambda_1^k(h_1^k)$ is the set of offers available to division d_1^k in round 1 and $\Lambda_n^k(h_n^k)$ is the set of offers available to division d_n^k in round n given the history of play h_n^k. Thus, the action λ_n^k is an element of $2^{\Lambda_n^k(h_n^k)}$, the action space for d_n^k. Finally, for any two distinct institutions k and k', I shall treat the internal choice games $\mathcal{G}(k)$ and $\mathcal{G}(k')$ as independent of each other.[22]

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[22]To be completely strict, an extensive game formalization of the second stage would require some specification of how rounds of an institution's internal choice game relates to the rounds of another's, and might therefore allow for the strategy of a division in one institution to depend on the choice of a division in another institution. The assumption

Given the list of actions a, where

$$a \equiv \left((\omega_i)_{i \in I}, \left(\left(\lambda_n^k \right)_{n=1}^{n=|D(k)|} \right)_{k \in K} \right),$$

the outcome of the game $\mathbb{G}(\mathcal{E})$ is a set of contracts $A(a) \equiv \bigcup_{k \in K} \bigcup_{n=1}^{n=|D(k)|} \lambda_n^k$. A strategy for a division $d_n^k \in D(k)$, denoted σ_n^k, is a map from the set of all possible histories at round n in the second stage, $H_n^k \equiv \{h_n^k\}$, to the *feasible* set of actions $\Lambda_n^k(h_n^k) \subseteq X(k)$. Let Σ_n^k be the set of all strategies for division d_n^k. A strategy for a talent i, denoted σ_i, is a map from $\prod_i \Omega_i$ to Ω_i. Let Σ_i be the set of all strategies for talent i. Define the strategy space Σ by

$$\Sigma \equiv (\Sigma_i)_{i \in I} \times \left(\left(\Sigma_n^k \right)_{n=1}^{n=|D(k)|} \right)_{k \in K}.$$

Every strategy profile $\sigma \in \Sigma$ induces a *path of play* $a(\sigma)$, which is a list of actions of each talent and division, and an *outcome* $A(\sigma) \equiv A(a(\sigma))$.

A strategy profile $\sigma \in \Sigma$ is a Subgame Perfect Nash Equilibrium (SPNE) if

- for every division d_n^k and every $\tilde{\sigma} \in \Sigma_n^k \times \{\sigma_{-d_n^k}\}$, it is the case that $A(\sigma) R^d A(\tilde{\sigma})$ at every history $h_n^k \in H_n^k$.

- for every talent i and every $\tilde{\sigma} \in \Sigma_i \times \{\sigma_{-i}\}$, it is the case that $A(\sigma) R^i A(\tilde{\sigma})$.

Since every list of talent offers induces a subgame for the divisions in each institution, we will first study the internal choice game induced by a particular list of offers $\omega \in \Omega$. The internal choice game induced by a hierarchical governance structure gives each division a unique weakly dominant strategy to choose at each realization of history its preference maximizing set of offers, taking ω as a parameter. Once ω is endogenized by embedding the internal choice game into the two-stage bargaining game, the unique weak dominance of this strategy remains. Denote this dominant strategy by $\hat{\sigma}_n^k$, where for any history $h_n^k \in H_n^k$,

$$\hat{\sigma}_n^k(h_n^k) = \max_{\mathcal{P}^{d_n^k}} \Lambda_n^k(h_n^k).$$

Moreover, requiring subgame perfection eliminates the use of any other strategy in equilibrium. Therefore, the divisions actions and the final outcome of the internal choice game $\mathcal{G}^k$ corresponds with the internal allocation and institutional choice produced by the inclusionary hierarchical procedure.

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of these internal choice games as being independent of each other is tantamount to analyzing a strict formalization with one division per round with a restriction of the class of strategies allowed. However, given the focus on subgame perfection, this restriction will not have a material impact on the equilibrium outcomes. An alternative formalization would be to model all institutional choice games occurring simultaneously, but with each choice game being sequential.

Lemma 2. *In any SPNE of $\mathbb{G}$, the strategy of any division d_n^k is $\hat{\sigma}_n^k$. For any SPNE σ^*, $\mathcal{G}^k(\omega)$ yields the outcome $C^k(\omega_k)$, where $\omega \equiv \prod_{i \in I} \sigma_i^*$.*

Proof. At any history $h \in H_n^k$, division d_n^k can determine its contracts in the outcome of the game by its choice from the available offers $\Lambda_n^k(h)$, no matter what subsequent actions are taken by other players. Therefore, the unique best response of d_n^k at history h is to choose the action of that corresponds to picking its preference-maximizing bundle from $\Lambda_n^k(h)$, which is exactly the prescribed action according to strategy $\hat{\sigma}_n^k$.

Since in SPNE every division takes the action of choosing its most preferred bundle of contracts, the outcome at this equilibrium coincides with the revelation mechanism induced by the institutional governance structure *qua* mechanism ψ^k given ω, which is strategyproof, and immediately yields the conclusion that the internal choice game $\mathcal{G}^k$ at ω reproduces the derived institutional choice function $C^k(\omega_k, \psi^k(\omega^k, (\mathcal{P}^d)_{d \in D(k)}))$, denoted $C^k(\omega_k)$ for simplicity, where $(\mathcal{P}^d)_{d \in D(k)}$ are the true preferences of divisions in $D(k)$. $\square$

The previous lemma justifies the reduction of the second stage in the subsequent propositions to a list of choice functions C^k. The interpretation is that with the inclusionary hierarchical governance, the internal game amongst divisions can be separated from the game between talent and institutions as a whole, given the focus on SPNE.

The first result will be to demonstrate pairwise stability of the outcome in SPNE. Note that the proof, and hence the result, does not require any assumption on preferences of divisions (and would only require the assumption of IRC on institutional choice if this choice is taken to be the primitive).

Proposition 5. *Let $\sigma^* \in \Sigma$ be an SPNE of the bargaining game $\mathbb{G}$ and let $a(\sigma^*)$ be the associated equilibrium actions and $A(\sigma^*)$ be associated equilibrium outcome. Then $A(\sigma^*)$ is pairwise stable.*

Proof. We know from lemma 2 that in SPNE, the subgame at any talent strategy profile ω, $\mathcal{G}^k(\omega)$ yields as the outcome the institutional choice function C^k derived from the inclusionary hierarchical procedure. That is, for any $(\sigma_i)_{i \in I} \in \prod_i \Sigma_i$, the outcome of the subgame at history $h_0 = (\omega)$ is exactly $C^K(h_0) \equiv \bigcup_{k \in K} C^k(\omega_k)$. The game $\mathbb{G}$ is thereby reduced to a simultaneous game amongst the talent.

Now, suppose that the SPNE outcome $A(\sigma^*)$ is not pairwise stable. Then there exists $i \in I$, $k \in K$ and $z \in X(i,k) \backslash A(\sigma^*)$ such that $z \in C^k(A(\sigma^*) \cup \{z\})$ and $z \in C^i(A(\sigma^*) \cup \{z\})$. Suppose talent i were to deviate from offering σ_i^* to offering z. Then, since C^k satisfies IRC, $z \in C^k(A(\sigma^*) \cup \{z\})$ and $\sigma_i^* \notin C^k(A(\sigma^*) \cup \{z\})$ implies $z \in C^k((A(\sigma^*) \cup \{z\}) \backslash \sigma_i^*)$, and so $z \in A((\tilde{\sigma}_i, \sigma_{-i}^*))$, where $\tilde{\sigma}_i = z$. But then i strictly prefers the outcome from playing $\tilde{\sigma}_i$ to playing σ_i^*, contradicting our assumption that σ^* is SPNE. Thus, $A(\sigma^*)$ is pairwise stable. $\square$

Subgame perfection is not strong enough to ensure stability of outcomes because talents can fail to "coordinate" with their proposed contracts, as described in the following example.

Example 2. Suppose there are two talents Ian i and John j and an institution Konsulting Group k. Let x and x' be two potential contracts between Ian and Konsulting, and let y and y' be two potential contracts between John and Konsulting. Imagine, perhaps, that contracts x and y stipulate working on the East Coast and contracts x' and y' stipulate working on the West Coast. Suppose Ian prefers the West Coast contract to the East Coast contract, as does John i.e. $x'\mathcal{P}^i x \mathcal{P}^i \emptyset$ and $y'\mathcal{P}^j y \mathcal{P}^j \emptyset$. Also, suppose that Konsulting Group is composed of just one division d, which would like to hire at least one of Ian or John in either geographical region, but does not want to hire both in *different* regions: $\{x', y'\}\mathcal{P}^d\{x, y\}\mathcal{P}^d x \mathcal{P}^d y \mathcal{P}^d x' \mathcal{P}^d y' \mathcal{P}^d \emptyset$. While other talents and institutions may be present, they are not required to demonstrate the "coordination failure" amongst talents; assume that no other talents are acceptable to Konsulting Group and that Ian and John are unacceptable to every other institution $k' \neq k$. Suppose in the non-cooperative bargaining game described above Ian offers only contract x and John offers only contract y, and suppose the one division in Konsulting Group chooses according to its preference, which it has a weakly dominant strategy to do. Then both x and y are chosen, and moreover are SPNE strategies for each talent, since Ian cannot improve by offering x' instead of (or as well as) x, given that John is offering only y, and vice versa. Notice also that the division's preferences satisfy bilateral substitutes, and that $\{x, y\}$ is pairwise stable but not stable. The only stable outcome is $\{x', y'\}$, which constitutes another SPNE outcome, supported for example by Ian offering x and John offering y. Both Ian and John prefer the equilibrium outcome $\{x', y'\}$ to $\{x, y\}$, but cannot unilaterally prevent the less-preferred outcome. In fact, even the division prefers $\{x', y'\}$ to $\{x, y\}$, and so SPNE outcomes can be inefficient.

When viewing institutional choice as primitive, stability of SPNE outcomes can be recovered by strengthening the assumptions on these choice functions. Suppose that every institution has a choice function satisfying IRC, bilateral substitutes and the Pareto Separable condition. Now, SPNE outcomes are stable and not just pairwise stable.

The power of the Pareto Separable condition comes from the property that the set of contracts between an institution and a talent now has a structure that is independent of the set of contracts with other talents available to the institution. A pair of contracts on which the institution and the talent have opposing choice behavior in some choice situation will never be harmonized in some other choice situation. This property is satisfied by substitutable choice, but is not a characteristic of it, since bilaterally substitutable choice functions that are not substitutable can still be Pareto Separable.

Proposition 6. *Suppose institutional choice functions are Pareto Separable and satisfy IRC and weak substitutes. Then every SPNE outcome is stable.*

The proof of the proposition lies in the recognition that under the assumption of bilateral substitutes and Pareto Separability, every group block can be reduced to an appropriate pairwise block, and thus every pairwise stable outcome is also stable. In fact, we can weaken the assumption from bilateral substitutability to weak substitutability, because these two substitutes conditions are equivalent given the Pareto Separable condition, stated in Proposition 1.

The equivalence of stability concepts under the Pareto Separable condition is the key lemma to the proof of stability of SPNE outcomes, and can be understood by recognizing that a block of an outcome that involves a contract between an institution and talent who have a contract with each other in the blocked allocation, a *renegotiation*, can be reduced to a block by just this contract. Similarly, any group block that does not have a renegotiation cannot involve more than one contract, if bilateral substitutability is to remain inviolate. But then any block can be reduced to a singleton block, and so stability is equivalent to pairwise stability.

Lemma 3. *Suppose institutional choice functions are Pareto Separable and satisfy IRC and weak substitutes. Then the set of stable outcome coincides with the set of pairwise stable outcomes.*

Proof. It is clear that every stable outcome is pairwise stable, by definition. To prove the converse, suppose A is pairwise stable. Assume that A is not stable. Then there exists an institution k and $Z \subseteq X(k) \backslash A(k)$ such that $Z \subseteq C^k(A \cup Z)$ and $Z(i) \mathcal{P}^i A(i)$ for every $i \in I(Z)$, and such that no $Z' \subsetneq Z$ has this same blocking property as Z. We say that such a Z is a *minimal* blocking group. We will show that $|Z| = 1$, contradicting the assumption that A is not pairwise blocked.

First, suppose that there exists $z \in Z$ such that the talent $I(z)$ has a contract with k in A i.e. $I(z) \in I(A(k))$. Let $y \in A(k)$ be the contract between $I(z)$ and k in A that is renegotiated via the block Z. Since $z \in C^k(A(k) \cup Z)$ and $y \in A(k)$, from the Pareto Separable condition we have that $y \notin C^k(A(k) \cup \{z\})$. Now, suppose $z \notin C^k(A(k) \cup \{z\})$. Then, by IRC we know that $C^k(A(k) \cup \{z\}) = C^k(A(k)) \ni y$, a contradiction. Thus, $z \in C^k(A(k) \cup \{z\})$, which implies that $\{z\}$ blocks A. Given that Z is a minimal blocking set, this implies $Z = \{z\}$ and so A is not pairwise stable, a contradiction.

Second, suppose that for every $z \in Z$, talent $I(z)$ does not have a contract with k in A i.e. $I(z) \notin I(A(k))$. Suppose that there exist $z, x \in Z$ where $z \neq x$. Clearly, $I(z) \neq I(x)$ given IRC and the assumption that a talent-institution pair can sign at most one contract in an allocation. Define $Y = A(k) \cup (Z \backslash \{z, x\})$. Since Z is a minimal block, $z \notin C^k(Y \cup \{z\}) = C^k(A(k))$ where the equality follows from IRC. However, $z \in C^k(Y \cup \{z, x\}) = C^k(A(k) \cup Z)$ by definition of a block. However, given that $I(z), I(x) \notin I(A(k))$ and since $|A(k)| = |I(A(k))|$, this block would violate assumption that C^k satisfies weak substitutes. Thus, Z must contain no more than one contract and so A is not pairwise stable, a contradiction.

Thus, we have proved that every pairwise stable outcome is stable. $\qquad\square$

Hence our proof of Proposition 6 is an immediate application of our previous results.

Proof. From Proposition 5 we have that every SPNE outcome is pairwise stable. From Lemma 3 we have that every pairwise stable outcome is stable. □

Another result is that the SPNE outcomes of the bargaining game are stable under the assumption that all divisions have substitutable preferences. Given the discussion of the previous section that substitutability of preferences of divisions does not ensure substitutability or even unilateral substitutability of institutional choice, this result proves stability of the noncooperative bargaining game outcomes for this class of bilaterally substitutable institutional choice functions. Note that the following proposition does not following from Proposition 6, because the property of Pareto Separability need not be preserved by inclusionary hierarchical procedures.

Proposition 7. *Suppose that every division has substitutable preferences. Then every SPNE outcome of the game* $\mathbb{G}$ *is stable.*

The proof of the proposition follows immediately given the following lemma.

Lemma 4. *Suppose every division has substitutable preferences. Then every pairwise stable outcome is stable.*

Proof. Let $A \subseteq X$ be a pairwise stable outcome. Suppose that there exists a blocking set $Z \subseteq X \backslash A$ involving institution k, so that $Z \subseteq C^k(A(k) \cup Z)$ and $z\mathcal{P}^{I(z)}A(I(z))$ for every $z \in Z$. Under the inclusionary hierarchical procedure, every contract in Z is allocated divisions in $D(k)$. Denote by f the internally stable allocation given choice situation $A(k)$ and by g the internally stable allocation given the choice situation $A(k) \cup Z$ i.e. $f \equiv \psi^k\left(A(k), (\mathcal{P}^d)_{d \in D(k)}\right)$ and $g \equiv \psi^k\left(A(k) \cup Z, (\mathcal{P}^d)_{d \in D(k)}\right)$. Let $\hat{d}$ be highest-ranked division to obtain one or more contracts from Z, define as follows: $Z \cap g(\hat{d}) \neq \emptyset$ and for every $d \rhd^k \hat{d}$, $Z \cap g(d) = \emptyset$. We will show that there exists some contract $\hat{z} \in Z$ such that $\hat{z}$ constitutes a pairwise block of A, contradicting the opening assumption.

Let $\hat{z} \in Z' \equiv Z \cap g(\hat{d}) \neq \emptyset$. By definition no division $d \rhd^k \hat{d}$ is allocated a contract in Z in choice situation $A(k) \cup Z$. Also, none of the talents with contracts in Z have alternative contracts in A that are allocated under g to any division higher-ranked than $\hat{d}$, since feasibility of the internal allocation would then prevent any such talent's contract in Z being chosen by the institution. We know that for every division $d \rhd^k \hat{d}$ $g(d) = f(d)$ by IRC of division choice, trivially satisfied since divisions have preferences. In fact, IRC yields another conclusion, that $g'(d) = f(d)$ for every $d \rhd^k \hat{d}$, where $g \equiv \psi^k\left(A(k) \cup \{\hat{z}\}, (\mathcal{P}^d)_{d \in D(k)}\right)$. Consider also that when the inclusionary hierarchical procedure determines the allocation from $A(k) \cup Z$ for $\hat{d}$, every contract that is available at this stage when the choice situation for the institution is $A(k)$, call it A', is still available for $\hat{d}$ in the expanded choice situation $A(k) \cup Z$. By IRC of division's choice, we know that removing contracts

in Z that are not in Z' has no effect on choice of $\hat{d}$. By substitutability of division's choice, we know that $\hat{z} \in C^{\hat{d}}(A' \cup Z')$ implies $\hat{z} \in C^{\hat{d}}(A' \cup \{\hat{z}\})$. But then $\hat{z} \in C^k(A(k) \cup \{\hat{z}\})$, and so $\hat{z}$ blocks A, which contradicts the assumption of pairwise stability of A, and concludes the proof. $\qquad\square$

An implementation result analogous to some in the literature, however, is not forthcoming, as the following example shows. The difficulty with achieving implementation in SPNE in a setting with multiple potential contracts between the two sides and with talents offering first is that there is very little competition over institutions, since talents do not make offers to more than one institution. This gives a lot of bargaining power to the talents, and makes it so that any bilateral "surplus" consistent with stability goes to the first mover, the talents.

Example 3. Suppose there is one institution k trivially consisting of one division d and three talents i_x, i_y, i_z, where the choice function of the division is given as follows:

$$
\begin{array}{lll}
Y \to C(Y) & Y \to C(Y) & Y \to C(Y) \\
\{x\} \to \{x\} & \{x,y\} \to \{x,y\} & \{x,y'\} \to \{x,y'\} \\
\{y\} \to \{y\} & \{x,z\} \to \{x,z\} & \{y,y'\} \to \{y'\} \\
\{z\} \to \{z\} & \{y,z\} \to \{y,z\} & \{y',z\} \to \{y',z\} \\
\{y'\} \to \{y'\} & & \\
\{x,y,z\} \to \{x,y,z\} & \{x,y',z\} \to \{x,y'\} & \{x,y,y',z\} \to \{x,y,z\}
\end{array}
$$

with contract x belonging to i_x, contracts y and y' to i_y, and contract z to i_z.

Suppose preferences of the three agents are: $xP^{i_x}\emptyset$, $yP^{i_y}y'P^{i_y}\emptyset$ and $zP^{i_z}\emptyset$. The choice function satisfies BLS and IRC, and is (for example) consistent with the following preferences:

$$
\{x,y,z\}\mathcal{P}^d\{x,y'\}\mathcal{P}^d\{y',z\}\mathcal{P}^d\{x,y\}\mathcal{P}^d\{y,z\}\mathcal{P}^d\{x,z\}\mathcal{P}^d\{y'\}\mathcal{P}^d\{y\}\mathcal{P}^d\{x\}\mathcal{P}^d\{z\}\mathcal{P}^d\emptyset
$$

for the division.

There is only one stable allocation $A_1 \equiv \{x,y,z\}$, which is also pairwise stable. However, $A_2 \equiv \{x,y'\}$ is also pairwise stable, though unstable.

Note that A_2 cannot be supported as a SPNE of the game $\mathbb{G}$, because t_y could strictly improve by offering y instead of y', keeping fixed the offers of other talents, which must be x by t_x and could be either z or $\emptyset$. If t_z is offering z, then if t_y offers y the division picks $\{x,y,z\}$. If t_z is offering $\emptyset$, then the division picks $\{x,y\}$. Thus, A_2 cannot be an SPNE outcome.

5 Conclusion

Stability has proven to be an important requirement that market outcomes should satisfy if the market is to function well. Using a matching-theoretic model, in this paper I show how hierarchies as a governance mode in institutions might persist in the market as a result of choice behavior that ensures stable market outcomes, a property that is not shared by some other organizational modes within institutions.

The novel approach complements existing theories for the presence of hierarchies in institutions in a market setting. Hierarchies induce institutionally efficient and strategyproof internal assignment rules while also producing market-level choice behavior that ensures stability. An important departure taken in this paper from the standard matching with contracts framework is that institutions are groups of decision-makers enjoined by a governance structure, which is modeled as an internal assignment rule. The decentralized market, studied as a noncooperative take-it-or-leave-it bargaining game, supports the conclusion that market outcomes will be pairwise stable generally, and stable under the assumption of substitutable preferences for divisions.

While the focus of this paper is on hierarchical governance within institutions, other governance structures could be considered, especially ones that allow for multiple internally stable assignments. Broadly speaking, the institutions could be thought of as competing allocation systems, with talents selecting into a particular institution. With the recent implementation of school choice mechanisms proposed by market designers by some school systems comes the scenario of geographically competing school choice mechanisms. For example, Washington DC has a voucher system for use in private schools, while simultaneously have a public school system with some scope for school choice. The fact that students can match across these two systems, and that each system has its own governance, means that stability *across* the two systems may not be guaranteed, though they may well be guaranteed within each system. Further research along this line of inquiry will prove valuable to market designers.

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A The Comparative Statics of Combinatorial Choice

Fix a choice function. For any set of contracts Y, let $R(Y)$ be the set of contracts rejected from Y and $C(Y)$ the set of contracts chosen from Y, and let $I(Y)$ be the set of talents with contracts in Y. Let A be the current set of contracts available, and let a be a contract not in A. Define $\hat{A} \equiv A \cup \{a\}$.

- The condition *NewOfferChosen* (NOC) is satisfied if and only if the following is true: $a \in C(\hat{A})$.

- The condition *NewOfferFromNewTalent* (NOFNT) is satisfied if and only if the following is true: $I(a) \notin I(A)$.

- The condition *NewOfferFromHeldTalent* (NOFHT) is satisfied if and only if the following is true: $I(a) \in I(C(A))$.

- The condition *NewOfferFromRejectedTalent* (NOFRT) is satisfied if and only if the following is true: $I(a) \notin I(C(A))$.

- The condition *RenegotiateWithHeldTalent* (RWHT) is satisfied if and only if the following is true: $\Big(\exists x \in R(A), x \in C(\hat{A}) \wedge I(x) \in I(C(A))\Big)$.

- The set $\mathcal{RRT}$ is the set of talents rejected at A but recalled at $\hat{A}$, excepting the talent making the new offer i.e. $\mathcal{RRT} \equiv (I(A)\backslash I(C(A))) \cap I(C(\hat{A}))$.

- The condition *RecallRejectedTalent* (RRT) is satisfied if and only if the following is true: $\Big(\exists x \in R(A), x \in C(\hat{A}) \wedge I(x) \notin I(C(A))\Big)$. Equivalently, RRT is satisfied if and only if $\mathcal{RRT} \neq \emptyset$.

- The set $\mathcal{RHT}$ is the set of talents held at A but rejected at $\hat{A}$, excepting the talent making the new offer i.e. $\mathcal{RHT} \equiv I(C(A)) \cap \Big(I(A)\backslash I(C(\hat{A}))\Big)$.

- The condition *RejectHeldTalent* (RHT) is satisfied if and only if the following is true: $\Big(\exists i \in I(C(A)), i \notin I(C(\hat{A}))\Big)$. Equivalently, RHT is satisfied if and only if $\mathcal{RHT} \neq \emptyset$.

- The condition *UnitarySet* (UnitS) is satisfied if and only the following is true: $|I(A)| = |A|$.

Let A be a subset of contracts and $a \notin A$, with $\hat{A} \equiv A \cup \{a\}$.

1. A choice function fails IRC if $\neg$*NewOfferChosen* and (*RejectHeldTalent* or *RecallRejectedTalent* or *RenegotiateWithHeldTalent*).

2. A choice function fails ParSep if *RenegotiateWithHeldTalent*.

3. A choice function fails ULS if *RecallRejectedTalent*.

4. A choice function fails BLS if *NewOfferFromNewTalent* and *RecallRejectedTalent*.

5. A choice function satisfies Subs if and only if it is never the case that *RenegotiateWithHeldTalent* or *RecallRejectedTalent* is true.

6. A choice function fails WS if (IRC or *UnitarySet*) and *NewOfferFromNewTalent* and *NewOfferChosen* and ¬*RenegotiateWithHeldTalent* and *RecallRejectedTalent*.

For a summary of these comparative statics results, see Table 1.

<div align="center"><i>New Offer Chosen:</i> $a \in C(A \cup \{a\})$</div>

		Recall Rejected Talent	*¬Recall Rejected Talent*
New Offer From New Talent: $I(a) \notin I(A)$	**Renegotiate With Held Talent**	Fails ParSep Fails ULS Fails BLS	Fails ParSep
	¬Renegotiate With Held Talent	Fails ULS Fails BLS IRC or UnitS $\implies$ Fails WS	

		Recall Rejected Talent	*¬Recall Rejected Talent*
New Offer From Held Talent: $I(a) \in I(C(A))$	**Renegotiate With Held Talent**	Fails ParSep Fails ULS	Fails ParSep
	¬Renegotiate With Held Talent	Fails ULS	

		Recall Rejected Talent	*¬Recall Rejected Talent*
New Offer From Rejected Talent: $I(a) \in I(A)\backslash I(C(A))$	**Renegotiate With Held Talent**	Fails ParSep Fails ULS IRC $\implies$ Fails BLS	Fails ParSep
	¬Renegotiate With Held Talent	Fails ULS IRC $\implies$ Fails BLS IRC $\implies$ Fails WS	

Table 1: Categorizing Choice Behavior where A is initially available and $a \notin A$ is a new contract offer

B Concepts of Stability

An allocation $A \in \mathcal{A}$ is **pairwise stable** (or **contractwise stable**) if it is individually stable and there does not exist a contract $x \in X\backslash A$ such that $x \in C^{K(x)}(A\cup\{x\})$ and $x \in C^{I(x)}(A\cup\{x\})$.

An allocation $A \in \mathcal{A}$ is **renegotiation-proof** if it is individually stable and there does not exist $k \in K$ and $Y \subseteq X(I(A(k)), k)\backslash A$ such that $Y \subseteq C^k(A \cup Y)$ and $Y(j) \in C^j(A \cup Y)$

for every $j \in I(Y)$. This notion of stability rules out allocations where an institution and some subset of agents with which it holds contracts have alternate contracts amongst themselves that they would all choose over their current contracts if available. Thus, renegotiation-proof allocations are intra-coalitionally efficient.

An allocation $A \in \mathcal{A}$ is **strongly pairwise stable** if it is individually stable, renegotiation-proof, and there does not exist an agent-institution pair $(i, k) \in I \times K$ that have no contract with each other in A i.e. $A \cap X(i, k) = \emptyset$, a contract $x \in X(i, k$, and a collection of contracts $Y \subseteq X(I(A(k)), k) \backslash A(k)$ such that $Y \cup \{x\} \subseteq C^{K(x)}(A \cup Y \cup \{x\})$ and $x \in C^{I(x)}(A \cup \{x\})$ and $Y(j) \in C^j(A \cup Y)$ for every $j \in I(Y)$. This notion of stability rules out blocks coming from an institution and agent without an existing relationship where the institution can renegotiate with some agents with which it has an existing relationship. It is an enjoining of the renegotiation-proof concept and of the pairwise stable concept.

Note that the strongly pairwise stable outcomes need not be stable, because a blocking set of contracts in the latter concept can include more than one agent that does not have a held contract with the blocking institution (where w.l.o.g. there is one blocking institution). However, if all divisions have choice functions that satisfy BLS and IRC, then every strongly pairwise stable outcome is also stable.

Proposition 8. *If choice functions satisfy BLS and IRC, then the strongly pairwise stable set is equivalent to the stable set.*

Proof. Every stable outcome is strongly pairwise stable, so we shall prove the converse, and do so by contradiction. Suppose A is strongly pairwise stable but not stable. Since it is not stable, there exists an institution k, a subset of talents $J \subseteq I$, and a collection of contracts $Z \subseteq X \backslash A$ where every contract in Z involves k and some talent in J and no two distinct contracts in Z name the same talent, such that for every $j \in J$, $Z(j) \mathcal{P}^j A(j)$ and $Z \subseteq C^k(A \cup Z)$. This set of contracts Z blocks A. Without loss of generality, let us suppose that Z is a *minimal* blocking set i.e. there does not exist $Z' \subseteq Z$ such that $Z' \subseteq C^k(A \cup Z')$. Given that A is strongly pairwise stable, we also know that there exists at least two talents $i_1, i_2 \in J$ who do not have contracts in A with institution k. Let $z_1 \equiv Z(i_1)$ and $z_2 \equiv Z(i_2)$, and define $Y \equiv Z \backslash \{z_1, z_2\}$. Since Z is a minimal blocking set, we know that $Y \cap C^k(A \cup Y) = \emptyset$ and $(Y \cup \{z_1\}) \cap C^k(A \cup Y \cup \{z_1\}) = \emptyset$, so $z_1 \notin C^k(A \cup Y \cup z_1)$. But since $Z \subseteq C^k(A \cup Z)$, it must be that $z_1 \in C^k(A \cup Z)$. However, implies that C^k violates bilateral substitutes, since z_1 and z_2 are contracts with distinct talents who do not have any contracts with k in $A \cup Y$, which is a contradiction. $\square$

This result is the counterpart to the well-known result on pairwise stability and stability under the assumption of substitutability, stated here for completeness.

Result 1. *In the classical matching model, the set of pairwise and strongly pairwise stable allocations is identical. Moreover, if choice functions satisfy substitutability and IRC, then the set of stable matchings and the set of pairwise stable matchings coincide, and these sets coincide with the strongly pairwise stable set and the renegotiation-proof set.*

The following propositions document that the strong pairwise stability concept in the domain of BLS and IRC divisional choice functions is distinct from the weaker concepts of pairwise stability and renegotiation-proofness.

Proposition 9. *If choice functions satisfy BLS and IRC, then the pairwise stable set is distinct from the renegotiation-proof set, which is distinct from the strongly pairwise stable set.*

Proof. Consider the following example with one institution and three agents, where the choice function of the institution is given as follows:

$$
\begin{array}{ccc}
Y \to C(Y) & Y \to C(Y) & Y \to C(Y) \\
x \to x & xy \to xy & xy' \to xy' \\
y \to y & xz \to xz & yy' \to y' \\
z \to z & yz \to yz & y'z \to y'z \\
y' \to y' & & \\
xyz \to xyz & xy'z \to xy' & xyy'z \to xyz
\end{array}
$$

Suppose preferences of the three agents are: $xP_x\emptyset$, $yP_yy'P_y\emptyset$ and $zP_z\emptyset$. The choice function satisfies BLS and IRC, and is (for example) consistent with the following preferences:

$$xyz \succ xy' \succ y'z \succ xy \succ yz \succ xz \succ y' \succ y \succ x \succ z \succ \emptyset$$

for the institution. The set of stable allocations is

$$\{\{x, y, z\}\},$$

the set of strongly pairwise stable allocations is

$$\{\{x, y, z\}\},$$

the set of renegotiation-proof allocations is the set of all individually stable allocations, and the set of pairwise stable allocations is

$$\{\{x, y, z\}, \{x, y'\}\}.$$

$\square$

Finally, I show by example that under a notion of substitutability weaker than BLS, the notion of Weak Substitutes introduced in Hatfield and Kojima (2008), the equivalence between strong pairwise stability and stability is broken.

Proposition 10. *If choice functions satisfy WeakSubs and IRC, then the strongly pairwise stable set is distinct from the stable set.*

Proof. Consider the following example with one institution and three agents, where the choice function of the institution is given as follows:

$$
\begin{array}{ccc}
Y \to C(Y) & Y \to C(Y) & Y \to C(Y) \\
x \to x & xy \to xy & xy' \to y' \\
y \to y & xz \to xz & yy' \to y' \\
z \to z & yz \to yz & y'z \to y' \\
y' \to y' & & \\
xyz \to xyz & xy'z \to y' & xyy'z \to xyz
\end{array}
$$

Suppose preferences of the three agents are: $xP_x\emptyset$, $yP_yy'P_y\emptyset$ and $zP_z\emptyset$. The choice function satisfies Weak Subs and IRC, though it fails BLS, and is (for example) consistent with the following preferences:

$$
xyz \succ y' \succ xy \succ yz \succ xz \succ y \succ x \succ z \succ \emptyset
$$

for the institution. The set of stable allocations is

$$
\{\{x, y, z\}\},
$$

the set of strongly pairwise stable allocations is

$$
\{\{x, y, z\}, \{y'\}\},
$$

the set of renegotiation-proof allocations is the set of all individually stable allocations, and the set of pairwise stable allocations is

$$
\{\{x, y, z\}, \{y'\}\}.
$$

$\square$

C   Proofs

Definition 8. *Given a combinatorial choice function C with domain X, define the Blair relation $\succsim^R$ as follows: for any $A \subseteq X$, $B \subseteq X$, $A \succsim^R B$ if $A = C(A \cup B)$. Let $\succ^R$ be the asymmetric component of $\succsim^R$.*

The proofs of the main results (Theorems 1, 2, 3 and Proposition 2) are obtained by a simple induction argument, given the results below.

For the following proofs, let C_1 and C_2 be choice functions defined on some domain X, where $I(x)$ is the talent associated with contract $x \in X$. Let $C_1 \rightarrowtail C_2$ denote the choice function derived from the inclusionary hierarchical procedure, where division 1 ranks higher than division 2.

Proposition 11. *Suppose C_1 and C_2 satisfy IRC. Then $C \equiv C_1 \rightarrowtail C_2$ satisfies IRC.*

Proof. Let $Y \subseteq X$ and $x \notin Y$ such that $x \notin C(\hat{Y})$, where $\hat{Y} \equiv Y \cup \{x\}$. Then $x \in C_1(\hat{Y})$ and so $C_1(\hat{Y}) = C_1(Y)$, since C_1 satisfies IRC. If $I(x) \in I(C_1(Y))$, then $x \notin \tilde{R}_1(\hat{Y})$ implying $\tilde{R}_1(\hat{Y}) = \tilde{R}_1(Y)$ and so $C_2(\tilde{R}_1(\hat{Y})) = C_2(\tilde{R}_1(Y))$. Thus, $C(\hat{Y}) = C_1(\hat{Y}) \cup C_2(\tilde{R}_1(\hat{Y})) = C_1(Y) \cup C_2(\tilde{R}_1(Y)) = C(Y)$, so IRC is satisfied in this case.

Instead, if $I(x) \notin I(C_1(Y))$, then $x \in \tilde{R}_1(\hat{Y})$. Now, since $x \notin C(\hat{Y})$, it must be that $x \in C_2(\tilde{R}_1(\hat{Y}))$ and since $\tilde{R}_1(\hat{Y}) = \tilde{R}_1(Y) \cup \{x\}$, IRC of C_2 implies $C_2(\tilde{R}_1(\hat{Y})) = C_2(\tilde{R}_1(Y) \cup \{x\}) = C_2(\tilde{R}_1(Y))$, implying $C(\hat{Y}) = C(Y)$ and establishing that C satisfies IRC.

$\square$

Proposition 12. *Suppose C_1 and C_2 satisfy SARP. Then $C \equiv C_1 \rightarrowtail C_2$ satisfies SARP.*

Proof. Assume that C violates SARP, in order to obtain a contradiction. Given that SARP implies IRC, we know that C_1 and C_2 satisfy IRC. Then from Proposition 11 we know that C satisfies IRC. Finally, from Alva (2012) we know that if C satisfies IRC it satisfies WARP. So, if C violates SARP but not WARP, there exists a sequence $X_1, \ldots, X_n, X_{n+1} = X_1$, with $n \geq 3$, such that $Y_{m+1} \succsim^R Y_m$ for all $m \in \{1, \ldots, n\}$ and $Y_{l+1} \succ^R Y_l$ for at least one l, where $Y_m \equiv C(X_m)$ and $\succsim^R$ is the previously defined Blair relation associated with C. To see the connection between the condition in the definition of SARP and the Blair relation, notice that the cycle condition for SARP requires $Y_m \subseteq X_{m+1}$. Now, by IRC we get $Y_{m+1} = C(X_{m+1}) = C(Y_{m+1} \cup Y_m)$, which means that $Y_{m+1} \succsim^R Y_m$.

Next, define $a_m \equiv C_1(X_m) = C_1(Y_m)$, where the latter equality follows from IRC, define $b_m \equiv C_2(\tilde{R}_1(X_m))$, where $\tilde{R}_1(X_m) \equiv \{x \in X_m : I(x) \notin I(C_1(X_m))\}$. Notice that $b_m = Y_m \backslash a_m$ and that $a_m \cap b_m = \emptyset$. Also, for any $Z \subseteq X$, $a_m \succsim_1^R Z$, where $\succsim_1^R$ is the Blair relation generated by C_1. Since $a_m \subseteq X_m$ and $a_m \subseteq X_{m+1}$, and $a_{m+1} \subseteq X_{m+1}$, we have that $a_{m+1} \succ_1^R a_m$ or $a_{m+1} = a_m$. However, given that C_1 satisfies SARP, we cannot have $a_{m+1} \succsim_1^R a_m$ for all m and $a_{l+1} \succ_1^R a_l$ for some l. Thus, for any m, $a_m = a_{m+1}$.

Now, define $Z_m \equiv \tilde{R}_1(X_m)$. Notice that $b_m \subseteq Z_m$. Moreover, since $a_m = a_{m+1}$ and $b_m \cap a_m = \emptyset$, we have that $b_m \cap a_{m+1} = \emptyset$ and so $b_m \subseteq Z_{m+1}$. However, this means $b_{m+1} \succsim_2^R b_m$, where $\succsim_2^R$ is the Blair relation generated by C_2. Given that C_2 satisfies SARP, an analogous argument to the one in the previous paragraph, given for C_1, applies here and allows us to conclude that $b_m = b_{m+1}$

41

for any m. But then $Y_m = Y_{m+1}$ for all m, contradicting our assumption of a choice cycle. Thus, $C \equiv C_1 \rightarrowtail C_2$ satisfies SARP.

$\square$

Proposition 13. *Suppose C_1 and C_2 satisfy IRC and WeakSubs. Then $C \equiv C_1 \rightarrowtail C_2$ satisfies IRC and WeakSubs.*

Proof. We have already proved that C satisfies IRC under the given assumptions.

Let $Y \subseteq X$ such that $|I(Y)| = |Y|$. Let $x \in X \backslash Y$ and $I(x) \notin I(Y)$ and $z \in X \backslash Y$, $z \neq x$, $I(z) \notin I(Y \cup \{x\})$. Suppose $z \notin C(Y \cup \{z\})$. If $x \notin C(\hat{Y})$, where $\hat{Y} \equiv Y \cup \{z, x\}$, then by IRC of C_1 and C_2, and hence of C, we have that $C(\hat{Y}) = C(Y \cup \{z\})$ implying $z \notin C(\hat{Y})$. Instead, suppose $x \in C(\hat{Y})$. Now, $z \notin C(Y \cup \{z\})$ implies $z \notin C_1(Y \cup \{z\})$. By IRC of C_1, $x \notin C_1(\hat{Y})$ implies $z \notin C_1(\hat{Y})$, so, given $I(z) \notin I(Y \cup \{z\})$, $z \in \tilde{R}_1(\hat{Y})$. If $x \in C_1(\hat{Y})$, then $x \notin \tilde{R}_1(\hat{Y})$. Moreover, by WeakSubs of C_1, for any $y \notin C_1(Y \cup \{z\})$, $y \notin C_1(\hat{Y})$. Thus, given that there is no more than one contract per talent in the available sets, if $y \in \tilde{R}_1(Y \cup \{z\})$, then $y \in \tilde{R}_1(\hat{Y})$. Thus, by WeakSubs and IRC of C_2, given that $z \notin C_2(\tilde{R}_1(Y \cup \{z\}))$, it must be that $z \notin C_2(\tilde{R}_1(\hat{Y}))$. Finally, if $x \notin C_1(\hat{Y})$, then $\tilde{R}_1(\hat{Y}) = \tilde{R}_1(Y \cup \{z\}) \cup \{x\}$ and so again IRC and WeakSubs of C_2 implies $z \notin C_2(\tilde{R}_1(\hat{Y}))$. Thus, C satisfies WeakSubs.

$\square$

Proposition 14. *Suppose C_1 and C_2 satisfy IRC and BLS. Then $C \equiv C_1 \rightarrowtail C_2$ satisfies IRC and BLS.*

Proof. We have already proved that C satisfies IRC under the given assumptions.

Let $Y \subseteq X$, $x, z \in X \backslash Y$, $I(x) \neq I(z)$, $I(x), I(z) \notin I(Y)$. Suppose $z \notin C(Y \cup \{z\})$. Define $\hat{Y} \equiv Y \cup \{z, x\}$.

In the first case, suppose $x \notin C(\hat{Y})$. Then $x \notin C_1(\hat{Y})$. Since $I(x) \notin I(Y \cup \{x\})$, $x \in \tilde{R}_1(\hat{Y})$. By IRC of C_1, $z \notin C_1(\hat{Y})$ and $I(z) \notin I(Y \cup \{z\})$ implies $z \in \tilde{R}_1(\hat{Y})$. Thus, $\tilde{R}_1(\hat{Y}) = \tilde{R}_z(Y \cup \{z\}) \cup \{x\} = \tilde{R}_1(Y) \cup \{z, x\}$. Now, we know that $z \notin C_2(\tilde{R}_1(Y \cup \{z\}))$ and so by BLS of C_2, $z \notin C_2(\tilde{R}_1(\hat{Y}))$. Thus, $z \notin C_1(\hat{Y}) \cup C_2(\tilde{R}_1(\hat{Y})) = C(\hat{Y})$, proving that C satisfies the BLS condition for this case.

In the second case, suppose $x \in C(\hat{Y})$. In the first subcase, suppose $x \in C_1(\hat{Y})$. By BLS of C_1, $z \notin C_1(\hat{Y})$. Since $I(z) \notin I(Y \cup \{x\})$, $z \in \tilde{R}_1(\hat{Y})$. Moreover, by BLS of C_1, if $y \in \tilde{R}_1(Y \cup \{z\})$ and $I(y) \notin C_1(Y \cup \{z\})$ then $y \in \tilde{R}_1(\hat{Y})$, keeping in mind that $I(y) \neq I(x)$. Thus, $\tilde{R}_1(\hat{Y}) \supseteq \tilde{R}_1(Y \cup \{z\})$ and $I(z)$ has only one contract in $\tilde{R}_1(\hat{Y})$. Now if for all $y \in \tilde{R}_1(\hat{Y}) \backslash \tilde{R}_1(Y \cup \{z\})$, we have that $y \notin C_2(\tilde{R}_1(\hat{Y}))$, then IRC implies $z \notin C_2(\tilde{R}_1(\hat{Y}))$. Instead, if $y \in C_2(\tilde{R}_1(\hat{Y}))$ then by IRC we have $y \in C_2(\tilde{Y} \cup \{y\})$, where $\tilde{Y} \equiv \tilde{R}_1(\hat{Y}) \backslash \{w \in \tilde{R}_1(\hat{Y}) : I(w) = I(y)\}$. But now, since $I(y) \neq I(\tilde{Y})$ and since $|\tilde{Y}(I(z))| = 1$, BLS of C_2 implies that $z \notin C_2(\tilde{Y} \cup \{y\})$ and so by IRC $z \notin C_2(\tilde{R}_1(\hat{Y}))$. Thus, $z \notin C(\hat{Y})$.

42

In the second subcase of the second case, suppose $x \notin C_1(\hat{Y})$. Since $x \in C(\hat{Y})$, it must be that $x \in C_2(\tilde{R}_1(\hat{Y}))$. By IRC of C_1, we have that $\tilde{R}_1(\hat{Y}) = \tilde{R}_1(Y \cup \{z\}) \cup \{x\} = \tilde{R}_1(Y) \cup \{z, x\}$. By BLS of C_2, we have $z \notin C_2(\tilde{R}_1(Y) \cup \{z\})$, implying $z \notin C_2(\tilde{R}(Y) \cup \{z, x\}) = C_2(\tilde{R}_1(\hat{Y}))$ and so $z \notin C(\hat{Y})$.

Having established that $z \notin C(\hat{Y})$ in every case, we have that C satisfies BLS.

$\square$