Journal of Economics and Business 66 (2013) 98–124

Contents lists available at SciVerse ScienceDirect

Journal of Economics and Business

Liquidity provision in a limit order book without adverse selection☆

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ARTICLE INFO

Article history:
Received 11 October 2011
Received in revised form 8 January 2013
Accepted 17 January 2013

JEL classification:
G12
G14

Keywords:
Market microstructure
Limit order markets
Liquidity
Private values

ABSTRACT

In this paper, we develop a dynamic model of a limit order market populated with liquidity traders who have only private values. We characterize and analyze the equilibrium order placement strategies of traders and the conditional execution probabilities of limit orders as a function of traders' liquidity demand and the state of the limit order book. We solve for the equilibrium of the model numerically, and analyze its properties by performing comparative dynamics analysis. Our analysis shows that changes in the steady state of the limit order book and optimal order placement strategies reflect corresponding changes in the trade-off between order execution risk and the size of potential trading gains. The equilibrium order flow depends on the current state of the limit order book since a trader's optimal trading strategy is largely affected by the time and price priorities of the existing limit orders in the book. We demonstrate how changes in the dispersion of traders' private values affect optimal trading strategies and conditional execution probabilities of limit orders. Our main result is that the dispersion in private values across traders has a significant impact on the stationary state of the equilibrium limit order book and the average bid–ask spread. A wider distribution of private values leads to more order placement at prices away from the consensus value, and therefore, to a larger bid–ask spread. Further, our numerical simulations show that extending the life span of limit orders reduces the average bid–ask spread observed in equilibrium. Finally, we find

☆ For helpful comments and discussions, I thank Thomas Chemmanur, Robert Taggart, Alan Marcus, Hassan Tehranian, Burton Hollifield, Alex Boulatov, John Wald, Mark Liu, and conference participants at the 2007 FMA meetings. Special thanks to an anonymous referee and to the editor, Ken Kopecky, for helpful suggestions. I am solely responsible for all errors and omissions.

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0148-6195/$ – see front matter
http://dx.doi.org/10.1016/j.jeconbus.2013.01.001

that the equilibrium percentage of market order submissions is also increasing in the dispersion in liquidity traders' private values.

1. Introduction

In a limit order market, buyers and sellers can submit an order of one of two types. A market order executes immediately the best price posted by previous limit orders. A limit order specifies a particular price for the order and specifies a promise to trade at that price. The limit order book is a list of all unexecuted limit orders. Traders provide liquidity by submitting limit orders and consume liquidity by submitting market orders. Many financial assets are traded in limit order books. There are many stock exchanges around the world where trading takes place completely (e.g., Euronext, Stockholm, Helsinki, Hong Kong, Shanghai, Tokyo, Toronto and various Electronic Communication Networks) or partially through electronic limit order books (NYSE, Nasdaq, London). Despite this prevalence of limit order markets, the theoretical literature on limit order markets is very small. Understanding the dynamic choice between limit orders and market orders is important because rational agents can optimally use different trading strategies depending on the state of the limit order book and their subjective beliefs about the value of financial assets that are traded in these markets. These different strategies, in turn, can generate significant effects on price impact, trading volume, bid–ask spreads, and the volatility of market prices.

The objective of this paper is to develop a new model of dynamic optimal order placement in a limit order market in order to better understand the economic trade-offs underlying the choice between limit orders and market orders by incorporating the dynamic nature of limit order markets. In our simple setting with symmetrically informed traders each of which has a private valuation of an asset, we focus on the trade-off between the price of an order and its execution probability that is essential to the analysis of traders' choice between limit and market orders. This basic trade-off between order price and execution probability can be summarized as follows. A trader can always obtain a larger probability of execution at the cost of a less favorable execution price away from the bid–ask spread, which can be interpreted as an implicit cost for demanding liquidity. By definition, a market order is a "limit order" with execution probability one and therefore has no execution risk at all. The motivation for trade results from agents' differences in their private valuations of the asset, which causes the agents to have differences in their incentives to provide or consume liquidity. Traders with more extreme private values are more impatient than traders with moderate private valuations that are close to the mean of the probability distribution of private values. In our model, there is no independently moving common value component and the average private value of traders is constant. Therefore, we abstract from the risk of being picked off (winner's curse).[1] Thus, we focus on the trade-off between price and execution probability in a limit order market and its effect on liquidity provision in an environment without adverse selection. After modeling the arrival of traders (sellers or buyers) in the market, we characterize and analyze the equilibrium order placement strategies of traders in terms of the state of the limit order book and the execution probabilities of limit orders. We solve for the equilibrium of our model numerically using several parameter specifications and theoretically investigate its properties by performing comparative dynamics analysis. In our model, limit orders last for a finite number of periods, and they cannot be modified or canceled after submission. We devise and implement a numerical algorithm of successive approximations to solve for the stationary Markov equilibrium of the model. The algorithm is based on mapping the liquidity demand/supply of traders into their subjective order execution probabilities. Imposing a monotonicity restriction as in Hollifield, Miller, Såndas, and Slive (2006), we then invert this mapping to derive the liquidity demand/supply of the traders with respect to the execution probabilities of orders at different prices. Using this approach

[1] Since a limit order involves a commitment to a price, it is exposed to unfavorable changes in the common value of the asset. This adverse selection risk is called the winner's curse risk or picking-off risk.

recursively, we find the fixed point of traders' liquidity demand/supply and the corresponding execution probabilities in a stationary equilibrium. Our approach lends itself to the characterization of the equilibrium order book in a discrete Markov-chain state representation. Therefore, we are able to analyze the interactions between transient changes in the state of the book and the order flow.

Our model yields several interesting new results about the evolution of limit order book in time, the bid–ask spread, and the effect of price priority and time priority rules on the optimal placement of limit orders. Our main finding is that the dispersion in private values across traders is a major factor determining the stationary state of the equilibrium limit order book, the bid–ask spreads, and the depth of the quotes in the limit order book. When the dispersion of agents' private valuations of the asset is small, we predict that submitting limit buy or sell orders at price quotes far from the middle point of the limit order book is less profitable. Since, in this case, agents (on both sides of the book) are more patient, their demand for liquidity is lower, i.e., their tendency to submit more aggressive limit orders and market orders is not strong. Dynamically, this implies that the future execution probabilities of more conservative limit orders submitted (on the other side of the book) in the current time period are lower. Hence, the expected returns to placing buy (sell) limit orders that are far below (above) an investor's own private valuation are lower even though potential gains from limit orders conditional on execution are high. On the other hand, as the dispersion in agents' private values increases, the number of impatient traders with higher liquidity demands increases. This makes it more profitable for other traders (with moderate private values) to place more conservative limit orders with larger potential profits, since the execution probability of these orders increases with the presence of more aggressive traders demanding liquidity. Hence, under market conditions with a large dispersion in liquidity traders' private valuations, the expected returns to placing buy (sell) limit orders that are far below (above) an investor's own private valuation are higher in equilibrium. Thus, our results show that a wider distribution of private values leads to more order placement at prices away from the consensus value, and therefore, to a larger bid–ask spread. The results of our numerical simulations also show that the equilibrium order flow depends on the current state of the limit order book in the sense that an agent's optimal trading strategy is largely affected by the time and price priorities of the existing limit orders in the book. Our model generates another prediction regarding the effect of increasing the life span of limit orders. We find that as the life span of a limit order increases from two to three periods in our model, the average bid–ask spread falls in equilibrium. Finally, our results also show that the equilibrium percentage of market order submissions is also increasing in the dispersion in traders' private values.

The empirical market microstructure literature documents evidence suggesting that traders follow order placement strategies that depend on the state of the market characterized by the limit order book. Using data on limit and market orders from the Paris Bourse, Biais, Hillion, and Spatt (1995) document the persistence of order flow and find that traders react by submitting limit orders in rapid succession when the bid–ask spread or the depth at the quotes is large. Hamao and Hasbrouck (1995) study the limit order book of the Tokyo Stock Exchange and also document persistence in order flow. Harris and Hasbrouck (1996) show that the profitability of limit and market orders varies with market conditions on the NYSE. Goldstein and Kavajecz (2000) document substantial shifts in the willingness of traders to place limit orders during extreme market movements in the New York Stock Exchange (NYSE). Sandås (2001) analyzes data from the Stockholm Exchange to test the empirical implications of the static model of Glosten (1994) and rejects them. Hollifield, Miller, and Såndas (2004) show that changes in the relative profitability of limit and market orders are important in explaining the empirical variation in order submission strategies in the Stockholm Stock Exchange. They empirically characterize and estimate optimal order strategies by a monotone function which maps the liquidity demand of the investors into their subjective execution probabilities. They find little evidence against the monotonicity restriction on the estimated trading strategy, which is the basis of our theoretical framework.[2]

[2] Hollifield et al. (2006) provide another empirical evidence of variation in liquidity supply and demand in the Vancouver Stock Exchange by also incorporating endogenous estimation of arrival rates of traders into their model. They find that traders' decision to supply or demand liquidity depends on whether it is scarce or abundant in the current state of the market. Their

The trade-offs related to price, execution probability and the risk of winner's curse form the basic framework of the theoretical literature on the choice between limit orders and market orders: see Cohen, Maier, Schwartz, and Whitcomb (1981), Kumar and Seppi (1993), Glosten (1994), Chakravarty and Holden (1995), Handa and Schwartz (1996), Rock (1996), Seppi (1997), Parlour (1998), Foucault (1999), Biais, Martimort, and Rochet (2000), Foucault, Kadan, and Kandel (2005), Wald and Horrigan (2005), Goettler, Parlour, and Rajan (2005, 2009), and Rosu (2009). These papers theoretically analyze prices, trading volumes, and efficiency in limit order markets. Among the theoretical studies listed above, only Parlour (1998), Foucault (1999), Foucault et al. (2005), Goettler et al. (2005), and Rosu (2009) analyze limit order trading in a dynamic setting. Parlour (1998) has a finite-horizon model where traders with private values can place orders either at an ask price A or at a bid price B. Foucault (1999) obtains closed-form solutions for the stationary equilibrium of his infinite-horizon dynamic model, and analyzes the equilibrium implications of the trade-off between price and execution probability, and winner's curse. However, in his model, limit orders expire after only one time period. Goettler et al. (2005) solve numerically for the stationary Markov perfect equilibrium in a dynamic limit order market. Their focus is on transaction costs, picking-off risk due to adverse selection, and the relationship between the consensus (fundamental) value of an asset and the characteristics of the limit order book. Unlike in the last two studies, there is no risk of limit orders being picked off (adverse selection risk) in our model, since the common value component is fixed. Therefore, the main contribution of our paper to the literature on limit order markets is to study liquidity provision in an environment without adverse selection.[3]

The paper is organized as follows. We outline our model in Section 2. In Section 3, we solve for the equilibrium of our model numerically and describe our solution algorithm. Next, we present an illustrative numerical example in Section 4. In Section 5, we present and discuss the results of our analysis of comparative dynamics. We describe the empirical implications of our model in Section 6. Section 7 concludes.

2. Model

This section presents the theoretical model we analyze in detail in later sections. First, we provide assumptions on the trading rules and trader preferences.

2.1. Description of the dynamic trading game

We consider the market for a single risky asset. Traders with different private valuations for the asset arrive sequentially in the market with an opportunity to trade. Agents can place an order to buy or sell one unit of the asset at a price chosen from the finite set

$$P \equiv \{p_1, \ldots, p_N\}, \tag{1}$$

where $p_i < p_{i+1}$ for any $i \in \{1, \ldots, N-1\}$. In our numerical simulations, we specifically set $N = 4$, with $p_1 = 30$, $p_2 = 31$, $p_3 = 32$, $p_4 = 33$.[4] The variable t refers to both the time period t when the order is submitted and to the agent whose turn it is to place an order at time t, where $t \in \{0, 1, \ldots\}$.

Upon arriving at the market, if the trader t decides to place an order for one unit of the asset, she determines the type of her order. Once an order has been submitted, it will either trade immediately

evidence suggests that if liquidity is scarce and therefore valuable, liquidity traders supply liquidity, but when its abundantly available, they prefer to demand it.

[3] Foucault et al. (2005) also consider a dynamic model of a limit order market motivated by traders who have differences in waiting costs, and they analyze bid–ask spread dynamics, market resiliency, effect of tick size, and time to execution. They require limit order traders to undercut existing orders, without the option to submit orders at or away from the quotes. Rosu (2009) presents a continuous time version of the Foucault et al. (2005) model, and endogenizes their undercutting result. Goettler et al. (2009) and Rosu (2010) extend their previous models to introduce asymmetric information.

[4] It is possible to model the price grid P to include more than 4 consecutive prices. For simplicity, we focus here on specifications with $N = 4$ in our numerical simulations.

(i.e., it is a market order) or enter the queue of unexecuted orders which is referred to as the limit order book. If a limit order is not executed in M periods after it has been submitted, it automatically expires at the end of the Mth period. In our model, we first analyze model specifications with $M=2$. Later, we also model cases with $M=3$, and analyze the effect of increasing M (from 2 to 3) to the limit order market equilibrium in our model. We assume that once a limit order is submitted, it cannot be canceled. At any time t, the limit order book consists of outstanding orders to buy and sell stock at some feasible prices. Note that the maximum number of outstanding limit orders at any given time t and the maximum depth at any given price quote p_i are both equal to M. The number of outstanding limit orders at given prices and the age of each order completely determines the state of the order book. An implication of sequential trading is that none of the existing limit orders can have the same age in a given state of the market. Given that limit orders can last up to M periods and the price set P is finite, it follows that the number of possible states of the limit order book is also finite. In our basic model with four price points ($N=4$) where the limit orders last for $M=2$ periods, the total number of states S is equal to 61. The state space is denoted by Ω and each unique state of the limit order book is denoted by ω_s. For example,

$$\omega_{51} = \{+2, 0, -1, 0\}, \tag{2}$$

corresponds to the state of the limit order book at time t when there is already a limit buy order at price p_1 submitted at time $t-2$ and a limit sell order submitted at price p_3 at time $t-1$. Suppose that this is the state of the market at time t and consider the states to which the order book can make a transition at time $t+1$ depending on the order strategy of the trader at time t.

1. Do not place an order or place a market sell order at price p_1: $\omega_{16} = \{0, 0, -2, 0\}$
2. Place a limit sell order at p_2: $\omega_{41} = \{0, -1, -2, 0\}$
3. Place a limit sell order at p_3[5]: $\omega_{24} = \{0, 0, (-1, -2), 0\}$
4. Place a limit sell order at p_4: $\omega_{61} = \{0, 0, -2, -1\}$
5. Place a limit buy order at p_1: $\omega_{33} = \{+1, 0, -2, 0\}$
6. Place a limit buy order at p_2: $\omega_{29} = \{0, +1, -2, 0\}$
7. Place a limit buy order at p_3: $\omega_1 = \{0, 0, 0, 0\}$

We can further explain the construction of the state space of the limit order book as follows. With $M=2$, there can be one of 7 possible order combinations at any given price quote p_i: (1) no orders denoted by 0; (2) one buy order with age 1 denoted by +1; (3) one sell order with age 1 denoted by -1; (4) one buy order with age 2 denoted by +2; (5) one sell order with age 2 denoted by -2; (6) two buy orders with ages 1 and 2 respectively, denoted by (+1,+2); (7) two sell orders with ages 1 and 2 respectively, denoted by $(-1, -2)$. We first start with the empty book state, which is denoted by ω_1. We then consider all states where there exist limit orders at only one price quote: out of the above 7 possible order combinations, 6 of them actually contain orders and each of them can be placed at $N=4$ different prices, which yields 24 states from ω_2 to ω_{25}. Next, we consider all states where there exist limit orders at two different price quotes: out of the 4 single-order combinations above (i.e., 2, 3, 4, and 5), there are 12 (4×3) possible two-way permutations of which only 6 are feasible in a limit order book: $\{+1, +2\}$, $\{+1, -2\}$, $\{-1, -2\}$, $\{+2, +1\}$, $\{+2, -1\}$, and $\{-2, -1\}$. Given a price grid with $N=4$ units, each of these 6 two-way permutations can be placed into the limit order book in 6 (two-way combinations of 4 prices) different ways, which yields 36 other states from ω_{26} to ω_{61}. In general, for any $N \geq 4$ and $M=2$, the number of states is given by[6]:

$$S = 1 + 6N + 6\frac{N(N-1)}{2}. \tag{3}$$

[5] The notation $(-1, -2)$ indicates that there are two limit sell orders outstanding at the same price quote with ages 1 and 2 respectively. For example, in state ω_{24}, there are two limit sell orders outstanding at price p_3. Similarly, to denote two limit buy orders outstanding at the same price quote, we use the notation $(+1, +2)$.

[6] For example, if $M=2$ and $N=5$, the number of states is given by $S = 1 + 6 \times 5 + 6 \times 10 = 91$.

Similarly, for any $N \geq 4$ and $M = 3$, the number of states is given by[7]:

$$S = 1 + 14N + 36\frac{N(N-1)}{2} + 24\frac{N(N-1)(N-2)}{6}. \quad (4)$$

We will refer to the state of the market at time t as s_t. Given the structure of the model, in certain states of the market, a trader can outbid the best quote, bid at the best quote, or underbid the best quote. The ask price is the lowest quoted price of existing limit sell orders at time t, and it is denoted a_t. If the sell side of the book is empty, $a_t = +\infty$. Similarly, the bid price, b_t, is the highest limit price of existing limit orders to buy at time t. If the buy side of the book is empty, $b_t = -\infty$. To denote the state dependence of bid and ask prices, we also use the notation $a(s_t)$ and $b(s_t)$ for the ask and the bid, respectively.

Let K and L be such that $p_K = \max\{b(s_t), p_1\}$ and $p_L = \min\{a(s_t), p_N\}$. The trader's decision in state s_t will be denoted with the decision variables $d_k^s(t)$, $d_l^b(t)$ for $k = K, \ldots, N$ and $l = 1, \ldots, L$. If $d_k^s = 1$ for some k where $p_k > b(s_t)$, then the trader submits a limit sell order at price p_k. If the trader places a market sell order at the bid price $b(s_t) = p_K$, such that $d_K^s(t) = 1$, then the order is immediately matched with the oldest outstanding limit order at the bid price $b(s_t)$. If $d_l^b(t) = 1$ for some l where $p_l < a(s_t)$, the trader submits a limit buy order at price p_l. If the trader places a market buy order at the ask price $a(s_t) = p_L$, such that $d_L^b(t) = 1$. If the trader does not submit any order, then $d_k^s(t) = 0$, $d_l^b(t) = 0$ for all k and l. As implied by these definitions, orders are first prioritized by price and then by submission time.

2.2. Trader preferences

We assume that the traders are symmetrically informed. The rationale for trading results from the different liquidity demands of the agents characterized by their private valuations of the asset. Depending on their private valuations, agents may want to supply or demand liquidity to the market or not to submit an order at all. This renders the market as a private value auction. Hence, the decision to trade and the choice of the type and the price of an order is endogenous.

The tth agent's private valuation for the asset is denoted u_t. We consider an i.i.d. uniform distribution for private values which is centered at the mid-point of the prices at which orders can be submitted. Thus, u_t is distributed independently and identically across agents uniformly with support $[A, B]$, where $A = p_1 - w$, $B = p_N + w$, and w is a positive constant. The private valuation u_t can be interpreted as the tth agent's preference for liquidity and captures her willingness to hold the asset. Even though there is no explicit common value component in this setting, we can think of the common value of the asset as being fixed at $(A + B)/2$, or equivalently at the middle of the price grid P, which is equal to $(p_1 + p_N)/2$.[8] Thus, in our limit order market setting without adverse selection (the risk of being picked off), the only motivation for trade is agents' differences in their private valuations of the asset.

All agents are assumed to be risk neutral and maximize their expected utility. Conditional on arriving in the market at state s_t, the expected payoff of a trader who submits an order crucially

[7] In this case, there are a total of 15 possible order combinations at a single price quote: (1) no orders denoted by 0; (2) one buy order with age 1 denoted by +1; (3) one sell order with age 1 denoted by −1; (4) one buy order with age 2 denoted by +2; (5) one sell order with age 2 denoted by −2; (6) one buy order with age 3 denoted by +3; (7) one sell order with age 3 denoted by −3, (8) two buy orders with ages 1 and 2 respectively, denoted by (+1,+2); (9) two sell orders with ages 1 and 2 respectively, denoted by (− 1, − 2); (10) two buy orders with ages 1 and 3 respectively, denoted by (+1,+3); (11) two sell orders with ages 1 and 3 respectively, denoted by (− 1, − 3); (12) two buy orders with ages 2 and 3 respectively, denoted by (+2,+3); (13) two sell orders with ages 2 and 3 respectively, denoted by (− 2, − 3); (14) three buy orders with ages 1, 2, and 3 respectively, denoted by (+1,+2,+3); (15) three sell orders with ages 1, 2, and 3 respectively, denoted by (− 1, − 2, − 3). Then, the first state is the empty limit order book. The number of all states where there exist limit orders at only one price quote is $14N$. The number of all states where there exist limit orders at two different price quotes is $36N(N-1)/2$. Finally, the number of all states where there exist limit orders at three different price quotes is $24N(N-1)(N-2)/6$.

[8] Note that it is only a semantic distinction to state that there is no common value and private values are uniformly distributed over $[A, B]$ or to state that the common value is fixed at $(A + B)/2$ and private values are uniformly distributed over $[-(B-A)/2, +(B-A)/2]$. We thank to an anonymous referee for suggesting this insightful interpretation of our model.

depends on the conditional execution probability of that order. Conditional execution probabilities for buy and sell orders at each price p_i at time t, in state s_t are denoted by $\Psi_i^b(s_t)$ and $\Psi_i^s(s_t)$, respectively. We will show below how these probabilities can be computed given a rule that monotonically maps a trader's liquidity demand into those execution probabilities. Suppose that a trader with valuation u submits a buy order at price p_i. Then her expected payoff conditional on the information at time t is equal to $\Psi_i^b(s_t)[u - p_i]$. Similarly, the conditional expected payoff of a trader with valuation u is equal to $\Psi_i^s(s_t)[p_i - u]$ when she submits a sell order at price p_i. The trader chooses $d_k^s \in \{0, 1\}$ for $k = K, \ldots, N$ and $d_l^b \in \{0, 1\}$ for $l = 1, \ldots, L$ to maximize

$$\sum_{k=K}^{N} d_k^s \Psi_k^s(s_t)[p_k - u] + \sum_{l=1}^{L} d_l^b \Psi_l^b(s_t)[u - p_l], \tag{5}$$

subject to the constraint:

$$\sum_{k=K}^{N} d_k^s + \sum_{l=1}^{L} d_l^b \leq 1.$$

2.2.1. Indifference threshold valuations

To find the optimal decision rule that maps a trader's private valuation u_t to order submission prices in state s_t, we first define some threshold valuations. For $i > 1$, the parameter $\theta_i^b(s_t)$ denotes the threshold valuation that makes a trader just indifferent between submitting a buy order at price p_i and submitting a buy order at price p_{i-1} in state s_t. Thus, for all $i > 1$ such that $p_i \leq a(s_t)$, a trader strictly prefers submitting a buy order at p_i to submitting a buy order at p_{i-1} if and only if trader t's private valuation u_t is greater than the threshold $\theta_i^b(s_t)$. Similarly, for $i < N$, $\theta_i^s(s_t)$ denotes the threshold valuation that makes a trader just indifferent between submitting a sell order at price p_i and submitting a sell order at price p_{i+1} in state s_t. The parameters $\theta_N^s(s_t)$ and $\theta_1^b(s_t)$ are the limits of the closed interval $[\theta_N^s(s_t), \theta_1^b(s_t)]$, in which a trader makes a no-order decision.

The following lemma builds an important link between conditional order execution probabilities $\{\Psi_i^b(s_t), \Psi_i^s(s_t)\}_{i=1}^N$ and traders' private valuations:

Lemma 1. *For two buyers with valuations u and u', $u' > u$, who optimally choose to submit buy orders at prices p_i^b and p_j^b, respectively, we have:*

$$(u - u')(\Psi_i^b - \Psi_j^b) \geq 0. \tag{6}$$

Similarly, for two sellers with valuations u and u', $u' > u$, who optimally choose to submit sell orders at prices p_i^s and p_j^s, respectively, we have

$$(u' - u)(\Psi_i^s - \Psi_j^s) \geq 0.$$

Proof. By optimality,

$$\Psi_i^b(u - p_i^b) \geq \Psi_j^b(u - p_j^b),$$

$$\Psi_j^b(u' - p_j^b) \geq \Psi_i^b(u' - p_i^b).$$

Multiplying the second inequality by −1 and adding and rearranging gives:

$$(u - u')(\Psi_i^b - \Psi_j^b) \geq 0.$$

The proof is symmetric for the sell side. □

Lemma 1, adapted into our setting from Hollifield et al. (2006), shows that execution probabilities of optimally placed orders must be monotone with respect to the traders' private valuation, which measures their liquidity demand. This means that the higher the private valuation of a limit order buyer

($u' > u$), the higher will be the execution probability of her limit buy order ($\Psi_j^b \geq \Psi_i^b$) since buyers with higher private values have a greater liquidity demand on the buy side. In fact, if a buying trader's private valuation exceeds some threshold, she will submit a market buy order with probability 1. Similarly, the lower the private valuation of a seller ($u < u'$), the higher will be the execution probability of her limit sell order ($\Psi_i^s \geq \Psi_j^s$) since sellers with lower private values will be more impatient to sell the asset and therefore have a greater liquidity demand on the sell side. Further, if a selling trader's private valuation is below some threshold, she will submit a market sell order with probability 1. Thus, according to Lemma 1, optimality requires that the mapping from valuations to execution probabilities is monotone. Traders with extremely low or high values of u_t have a higher willingness to trade the asset immediately, whereas traders with moderate values of u_t demand liquidity more patiently and only if the current state of the order book presents them profitable trading opportunities. The next proposition shows that, as the limit order price moves away from the bid–ask spread, the conditional order execution probability decreases monotonically.

Proposition 1. *Suppose that there exist two agents who optimally submit buy orders at prices p_i^b and p_j^b respectively, and $\Psi_i^b \leq \Psi_j^b$. Then, it follows that $p_i^b \leq p_j^b$. Similarly, if p_i^s and p_j^s are some optimal prices to sell and $\Psi_i^s \leq \Psi_j^s$, it follows that $p_i^s \geq p_j^s$.*

Proof. Suppose that $p_i^b > p_j^b$. Since it is optimal for some agent to submit a buy order at p_i^b, optimality requires that there exists a u such that $u > p_i^b > p_j^b$. Then for any such u, we have

$$\Psi_j^b(u - p_j^b) > \Psi_i^b(u - p_i^b), \tag{7}$$

which implies that the agent with private value u will prefer to submit a buy order at p_j^b rather than p_i^b, hence a contradiction. The proof is symmetric for the sell side. □

Together with Lemma 1, Proposition 1 implies the monotonicity of the mapping from private valuations to order prices that could be optimal for some trader given the state of the limit order book. From Lemma 1, we know that for two buyers with valuations u and u' (where $u' > u$), who optimally choose to submit buy orders at prices p_i^b and p_j^b respectively, it follows that $\Psi_i^b \leq \Psi_j^b$. Further, Proposition 1 shows that in this case, the price p_j^b at which the trader with the higher private valuation u' submits his optimal buy order is not less than the price p_i^b at which the buying trader with the lower private valuation u submits her optimal buy order, i.e., $p_j^b \geq p_i^b$. Given the monotonicity of optimal order submissions in trader valuations, the optimal order placement strategy is fully characterized by the following proposition, which is also adapted to our model from Hollifield et al. (2006).

Proposition 2 (Optimal order placement strategy). *Suppose that a trader with private valuation u arrives at the market at time t and the state of the market is $s_t \in \Omega$. Trading opportunities in the limit order book are characterized by the conditional execution probabilities $\Psi_k^s(s_t)$ and $\Psi_l^b(s_t)$ for sell order choices $k = K, \ldots, N$ and buy order choices $l = 1, \ldots, L$, respectively. The set of prices that could be optimal for some trader given the state of the book are $p_K^s < \cdots < p_N^s$ on the sell side, where $p_K^s \geq b(s_t)$, with $p_L^b > \cdots > p_1^b$ defined similarly for the buy side, where $p_L \leq a(s_t)$. Defining*

$$\theta_k^s(s_t) = p_k - \frac{(p_{k+1} - p_k)\Psi_{k+1}^s(s_t)}{\Psi_k^s(s_t) - \Psi_{k+1}^s(s_t)}, \quad k = K, \ldots, N-1; \tag{8}$$

$$\theta_l^b(s_t) = p_l + \frac{(p_l - p_{l-1})\Psi_{l-1}^b(s_t)}{\Psi_{l-1}^b(s_t) - \Psi_l^b(s_t)}, \quad l = 2, \ldots, L; \tag{9}$$

$$\theta_1^b(s_t) = \theta_N^s(s_t) = \frac{\Psi_1^b(s_t)p_1 + \Psi_N^s(s_t)p_N}{\Psi_1^b(s_t) + \Psi_N^s(s_t)}. \tag{10}$$

The optimal decision rule is given by

$$d_K^{*s}(s_t) = \begin{cases} 1, & u \le \theta_K^s(s_t) \\ 0, & \text{otherwise} \end{cases}, \tag{11}$$

$$d_k^{*s}(s_t) = \begin{cases} 1, & u \in (\theta_{k-1}^s(s_t), \theta_k^s(s_t)] \\ 0, & \text{otherwise} \end{cases}, \quad k = K+1, \ldots, N; \tag{12}$$

$$d_l^{*b}(s_t) = \begin{cases} 1, & u \in (\theta_l^b(s_t), \theta_{l+1}^b(s_t)] \\ 0, & \text{otherwise} \end{cases}, \quad l = 1, \ldots, L-1; \tag{13}$$

$$d_L^{*b}(s_t) = \begin{cases} 1, & u > \theta_L^b(s_t) \\ 0, & \text{otherwise} \end{cases}. \tag{14}$$

The threshold valuations $\theta_k^s(s_t)$ and $\theta_l^b(s_t)$ given in Eqs. (8) and (9) respectively are obtained by solving for private valuations that make a trader just indifferent between submitting an order at two adjacent prices:

$$\Psi_k^s(s_t)(p_k - \theta_k^s(s_t)) = \Psi_{k+1}^s(s_t)(p_{k+1} - \theta_k^s(s_t)), \tag{15}$$

$$\Psi_l^b(s_t)(\theta_l^b(s_t) - p_l) = \Psi_{l-1}^b(s_t)(\theta_l^b(s_t) - p_{l-1}). \tag{16}$$

Lemma 1 and Proposition 1 together imply that traders' optimal order placement strategies are monotone in the following sense. For instance, when $a_t = p_L$ and all feasible buy order prices $p_L > \cdots > p_1$ are optimal for some buyers, then the corresponding threshold valuations form a monotone sequence $\theta_L^b(s_t) > \cdots > \theta_1^b(s_t)$ that divides the valuation line into $L+1$ intervals. The optimal decision rule maps these valuation intervals into bidding strategies at L different prices. Thus, traders with higher valuations submit buy orders with higher prices, which have a higher execution probability. Symmetric arguments apply to the sell side. Hence, traders with lower valuations submit sell orders with lower prices, which have a higher execution probability.

For a buyer indifferent between submitting a buy order at the lowest possible price p_1 and not entering an order, the threshold value $\theta_1^b(s_t)$ solves:

$$\Psi_1^b(s_t)(\theta_1^b(s_t) - p_1) = 0. \tag{17}$$

Thus, $\theta_1^b(s_t) = p_1$. Similarly, for the sell side, the indifference equation $\Psi_N^s(s_t)(p_N - \theta_N^s(s_t)) = 0$ implies that $\theta_N^s(s_t) = p_N$. But then, $[\theta_N^s, \theta_1^b] = [p_N, p_1]$ is an empty set (since $p_N > p_1$ by assumption), which shows that in the absence of a time-varying common value of the asset or trading cost differentials, the no-order decision is dominated in every state of the market for all agents.[9] In this case, if selling at price p_N and buying at price p_1 are not dominated, θ_N^s is identically equal to θ_1^b, and therefore, denotes the valuation at which the trader is indifferent between submitting a sell order at price p_N and a buy order at price p_1. This follows from the following indifference equation:

$$\Psi_1^b(s_t)(\theta_N^s(s_t) - p_1) = \Psi_N^s(s_t)(p_N - \theta_N^s(s_t)). \tag{18}$$

A representation of a trader's optimal order placement strategy as a monotone function of her private valuation is shown in a graphical example depicted in Fig. 1.

3. Solving for the equilibrium

In this section, we explain the numerical solution method that we implement to solve for the stationary Markov equilibrium of our dynamic limit order trading game.

[9] In our setting, an order placement strategy is dominated if there exists no agent with some private value $u \in [A, B]$ such that it is optimal for that agent to follow that strategy.

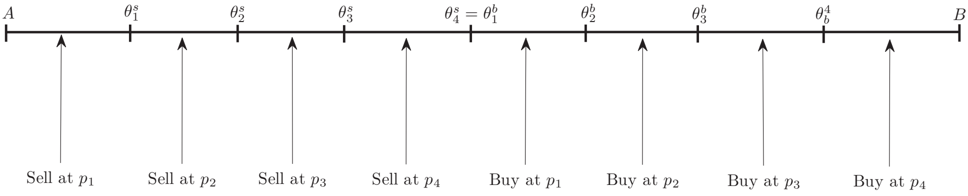

Fig. 1. Optimal order placement strategy as a function of the trader's private valuation. This figure presents a visualization of the optimal order strategy of a trader arriving at time t as a function of her private valuation of the asset. The sell threshold valuations $\theta_k^s(s_t)$ for $k = K, \ldots, N$ and the buy threshold valuations $\theta_l^b(s_t)$ for $l = 1, \ldots, L$ form a monotone sequence, which partitions the support $[A, B]$ of the private value distribution into the trader's optimal order choices. In this example, the price grid has $N = 4$ units, and the limit order book is in a state s_t, where $K = 1$ and $L = 4$.

3.1. Algorithm

The algorithm we implement to solve for the equilibrium is based on traders' optimal order placement strategy derived in Proposition 2 and the link between optimal order choice rules and the conditional execution probabilities of submitted orders. It solves for the fixed point of the correspondence on the space of optimal order placement rules defined by the monotone sequence of threshold valuations we explained in the previous section. Let Θ be the space of optimal order placement decision rules and $\theta(\Psi(s_t)) = \{\theta_i^b(s_t), \theta_j^s(s_t); i = 1, \ldots, L_{s_t}; j = K_{s_t}, \ldots, N\}$ be an arbitrary element of Θ implied by the conditional execution probabilities $\Psi(s_t)$ according to Proposition 2. Note that $\theta(\Psi(s_t))$ is a mapping defined on the trading opportunities $\Psi(s_t) = \{\Psi_i^b(s_t), \Psi_i^s(s_t); i = 1, \ldots, L_{s_t}; j = K_{s_t}, \ldots, N\}$ available in state s_t at time t. But once an optimal decision rule $\theta(\Psi(s_t))$ is specified, it also directly affects the conditional execution probabilities of the submitted orders. In other words, the trading opportunities defined by the conditional execution probabilities in a given state are also a function of the specific decision rule implemented and therefore, we can denote them as $\Psi(\theta(s_t))$. Thus, in a stationary Markov equilibrium of our model, the sequence of optimal order placement rules $\theta^n(\Psi^n(s_t))$ and the trading opportunities $\Psi^n(\theta^{n-1}(s_t))$ will converge to their stationary limits $\theta^*(\Psi^*(s_t))$ and $\Psi^*(\theta^*(s_t))$ respectively.

To understand the derivation of conditional order execution probabilities Ψ implied by an optimal decision rule θ, one should first note that an optimal order placement rule $\theta(s_t)$ determines the probabilities of all feasible order submissions in any given state s_t. Suppose that a trader t arrives in the market in a state s_t such that the sell side of the book is not empty, i.e., $a(s_t) < +\infty$. Using the optimal order placement strategy given in Proposition 2 and recalling that each trader's private valuation is uniformly distributed over the interval $[A, B]$ with cumulative distribution function F, the conditional probability that we will observe a market buy order is then equal to

$$P(d_L^{*b}(s_t) = 1) = P(u > \theta_L^b(s_t)) = 1 - F(\theta_L^b(s_t)). \tag{19}$$

The conditional probability that a limit buy order is submitted at time t at a particular price p_l, $l = 1, \ldots, L-1$ is equal to

$$P(d_l^{*b}(s_t) = 1) = P(\theta_l^b(s_t) < u \leq \theta_{l+1}^b(s_t)) = F(\theta_{l+1}^b(s_t)) - F(\theta_l^b(s_t)). \tag{20}$$

One can also obtain the conditional order submission probabilities for all feasible sell orders in state s_t similarly.[10] Thus, an optimal trading strategy $\theta \in \Theta$ monotonically maps traders' private valuations into their optimal order submissions and therefore, into conditional order submission probabilities across feasible price quotes in any state s_t.

Second, one should also note that since there are finitely many states of the limit order book, the transition possibilities among these states are well defined and finite. Given an order placement rule

[10] Recall that once an optimal trading strategy $\theta \in \Theta$ is specified, the no-order decision is dominated in our setting without adverse selection. Therefore, the probability of no-order submission is zero.

$\theta \in \Theta$, conditional order submission probabilities obtained in Eqs. (19) and (20) allow us to compute the one-period state transition probabilities between any two states in Ω. If a stationary equilibrium exists, we know that $\theta^* \in \Theta$ will also be the optimal decision rule of the traders arriving at times $t+1$ through $t+M$, since the underlying probability distribution of their private valuations is independently and identically distributed as that of the trader arriving at t. Thus, given that the decision rule $\theta \in \Theta$ also maps the private valuations of the traders in the following periods $t+1$ through $t+M$ to their optimal trading strategies and that limit orders last for M periods, we can then compute the conditional execution probabilities Ψ of all feasible limit orders that can be submitted at time t in any state $s_t \in \Omega$. We will illustrate this point via a numerical example (where $M=2$) in the next section.

Initially, the algorithm starts with an arbitrary decision rule or order placement strategy $\theta^o(s_t) = \{\theta_i^{b,o}(s_t), \theta_i^{s,o}(s_t), i = 1, \ldots, N\}_{s_t \in \Omega}$. This initial trading strategy consists of a monotone sequence of indifference valuations and the associated decision rule defined in Proposition 2.[11] We assume that initially, none of the feasible sell order prices $p_K^s < \cdots < p_N^s$ and feasible buy order prices $p_L^b > \cdots > p_1^b$ are dominated at the very beginning of the algorithm, i.e., for any feasible buy or sell order price in any state s_t, there exists a trader with private value u for whom it is optimal to place that order in state s_t.[12]

The next step of the algorithm is to determine the conditional execution probability of any feasible limit order at any given state, $\Psi^1(s_t) = \{\Psi_i^{b,1}(s_t), \Psi_j^{s,1}(s_t), i = 1, \ldots, L_{s_t}; j = K_{s_t}, \ldots, N\}$ given the initial trading strategy $\theta^o(s_t)$. Thus, in general, at the nth iteration of the algorithm, this step involves the determination of the trading opportunities $\Psi^n(s_t, \theta^{n-1}(s_t))$ (in any state s_t) implied by the order placement strategies $\theta^{n-1}(s_t)$ derived at the previous $(n-1)$th iteration. After finding $\Psi^1(s_t)$ $(\Psi^n(s_t))$ for all $s_t \in \Omega$, the next step is to determine the optimal trading strategy $\theta^1(s_t)$ $(\theta^n(s_t))$ for all $s_t \in \Omega$ as described in Proposition 2. This step of the algorithm requires in many cases (states of the market) the iterated elimination of some dominated order choices which the trader would never find optimal in state s_t regardless of her private valuation u. This process of iteratively eliminating the dominated order choices reduces the set of feasible buy and sell order choices in a state s_t to a set of price quotes and their associated order types, which an agent can find optimal in that state of the market given her private valuation u and the trading opportunities $\Psi^n(s_t)$.

These steps constitute a single iteration of the algorithm. They are then iterated until the optimal trading strategy $\theta^n(s_t)$ and the trading opportunities $\Psi^n(s_t)$ converge to some stationary limit points $\theta^*(s_t)$ and $\Psi^*(s_t)$, respectively, for all states $s_t \in \Omega$.

4. An illustrative example

Suppose that the private valuation u of any trader t is drawn independently from the uniform distribution with support [29, 34], and the price set (with $N=4$) is equal to $P=\{p_1=30, p_2=31, p_3=32, p_4=33\}$. Each unexecuted limit order can last up to $M=2$ periods. Assume that upon arriving in the market at time t, the trader t finds the limit order book empty, i.e., $s_t=\omega_1=\{0, 0, 0, 0\}$. Given the underlying probability distribution of a trader's private valuation and rationally expecting that any trader will use the same initial decision rule $\theta^o(s_t)$ given by

$$\{\theta_1^{s,o} = 29.5, \theta_2^{s,o} = 30, \theta_3^{s,o} = 30.5, \theta_4^{s,o} = 31, \theta_1^{b,o} = 32, \theta_2^{b,o} = 32.5, \theta_3^{b,o} = 33, \theta_4^{b,o} = 33.5\}$$

[11] The initial trading strategy $\theta^o(s_t)$ is not necessarily a member of the space of the optimal order placement decision rules Θ. In other words, different from the construction in Proposition 2, $\theta^o(s_t)$ is not derived from the available trading opportunities in the market characterized by a set of conditional execution probabilities $\Psi_k^{s,o}(s_t)$, $\Psi_l^{b,o}(s_t)$. Although we denote this initial sequence $\theta^o(s_t)$ as a function of the state s_t, it can be state independent as well, i.e., $\theta^o(s_t)=\theta^o$ for all $s_t \in \Omega$. In fact, we first tested our algorithm by using various state independent rules and then switched to randomized state dependent initial rules in order to test the robustness of our convergence results. We verified that the final results obtained after the algorithm converges to the steady state do not depend on the initial trading strategy.

[12] At the first iteration, one can also specify a no-order region, where $\theta_1^{b,o}(s_t) > \theta_N^{s,o}(s_t)$, although it follows from Proposition 2 that the no-order strategy is dominated in our setting with no winner's curse risk. Therefore, no-order placement is not a part of optimal trading strategies in subsequent iterations.

for all $s_t \in \Omega$, she can infer the conditional execution probabilities of all feasible orders she could submit in each state s_t, which are denoted by $\Psi^1(\theta^o(s_t)) = \{\Psi_i^{b,1}(s_t), \Psi_j^{s,1}(s_t)\}$, where $i = 1, \ldots, L_{s_t}$ and $j = K_{s_t}, \ldots, 4$.

We will illustrate this simple computation only for $\Psi_3^{b,1}(s_t)$, which is the conditional execution probability of a limit buy order submitted at price $p_3 = 32$, when the limit order book is in state $\omega_1 = \{0, 0, 0, 0\}$. After submitting this order at time t, the limit order book will be in the following state at time $t+1$:

$$s_{t+1} = \omega_4 = \{0, 0, +\mathbf{1}, 0\}. \tag{21}$$

Then, the probability that this limit buy order is executed at time $t+1$ (matched by a market sell order submitted at p_3) is equal to $F(\theta_3^{s,o}) = F(30.5) = (30.5 - 29)/(34 - 29) = 0.3$, given that trader $t+1$ will use the trading strategy θ^o. Then, with probability 0.7, the outstanding limit order at price p_3 will not be executed at time $t+1$. Consequently, it can be executed at time $t+2$ only in the following cases[13]:

1. Trader $t+1$ submits a limit buy order at p_1 with probability $F(\theta_2^{b,o}) - F(\theta_1^{b,o}) = (32.5 - 32)/(34 - 29) = 0.1$ and $s_{t+2} = \omega_{27} = \{+1, 0, \mathbf{+2}, 0\}$.
2. Trader $t+1$ submits a limit buy order at p_2 with probability $F(\theta_3^{b,o}) - F(\theta_2^{b,o}) = (33 - 32.5)/(34 - 29) = 0.1$ and $s_{t+2} = \omega_{29} = \{0, +1, \mathbf{+2}, 0\}$.
3. Trader $t+1$ submits a limit buy order at p_3 with probability $F(\theta_4^{b,o}) - F(\theta_3^{b,o}) = (33.5 - 33)/(34 - 29) = 0.1$ and $s_{t+2} = \omega_{20} = \{0, 0, (+1, \mathbf{+2}), 0\}$.
4. Trader $t+1$ submits a limit sell order at p_4 with probability $F(\theta_4^{s,o}) - F(\theta_3^{s,o}) = (31 - 30.5)/(34 - 29) = 0.1$ and $s_{t+2} = \omega_{55} = \{0, 0, \mathbf{+2}, -1\}$.
5. Trader $t+1$ does not submit any order with probability $F(\theta_1^{b,o}) - F(\theta_4^{s,o}) = (32 - 31)/(34 - 29) = 0.2$ and $s_{t+2} = \omega_{12} = \{0, 0, \mathbf{+2}, 0\}$.

In all these five cases, the bid price $b(s_{t+2})$ is equal to p_3 at time $t+2$, and the conditional probability of execution at time $t+2$ is therefore equal to the probability that trader $t+2$ submits a market sell order at p_3, which is equal to $F(\theta_3^{s,o}) = F(30.5) = 0.3$. Thus, given the decision rule θ_o and by backward induction, the execution probability of a limit buy order submitted at price p_3 at time t in state $s_t = \omega_1 = \{0, 0, 0, 0\}$ is equal to

$$\begin{aligned} \Psi_3^{b,1}(s_t) &= P(\text{execution at time t} + 1) + P(\text{execution at time t} + 2) \\ &= 0.3 + (4 \times 0.1 \times 0.3 + 0.2 \times 0.3) = 0.48. \end{aligned} \tag{22}$$

The same logic can be applied to find the conditional execution probabilities Ψ^n of any feasible order in any state implied by the agents' trading strategy θ^{n-1}.[14]

Next we consider an example for the next step of the algorithm that maps the agent's private valuation u into her optimal trading strategy $\theta^n(\Psi^n(s_t))$ given the trading opportunities $\Psi^n(s_t)$ available in state s_t. Now, we assume that upon arriving in the market at time t, the trader t finds the book in state $\omega_{51} \in \Omega$, i.e., $\omega_{51} = \{+2, 0, -1, 0\}$, where $a(\omega_{51}) = p_3$ and $b(\omega_{51}) = p_1$. Then, the trader infers the trading opportunities $\Psi^{s,1}(s_t) = \{1.00, 0.48, 0.16, 0.07\}$ and $\Psi^{b,1}(s_t) = \{0.18, 0.36, 1.00, 0\}$ as shown above. Note that in state ω_{51}, the agent cannot submit a buy order at p_4 because the ask price is equal to p_3, so we set $\Psi_4^{b,1} = 0$.

Given the conditional execution probabilities $\Psi^1(s_t)$ in state ω_{51}, our algorithm initially assumes that all feasible sell order choices at prices $p_1^s, \ldots, p_4^s$ and buy order choices at prices $p_1^b, \ldots, p_3^b$ can be optimal for some traders, and applies the decision rules given in Proposition 2 to find a

[13] Trader $t+1$ can also submit a limit buy order at p_4. But if this occurs, the existing limit buy order at p_3 (submitted at time t) has zero probability of execution at time $t+2$ due to the price priority rule.

[14] This line of analysis can be also generalized to the case where limit orders last for M periods where $M > 2$. In this paper, we also analyze the case when $M = 3$.

monotone sequence of threshold valuations $\theta^1(\Psi^1(s_t))$ that characterizes the optimal trading strategy of the trader as a function of her private value. This yields the following sequence in our example:

$$\{\theta_1^{s,1} = 29.08, \theta_2^{s,1} = 30.50, \theta_3^{s,1} = 31.22, \theta_4^{s,1} = 30.84, \theta_1^{b,1} = 30.84, \theta_2^{b,1} = 32.00, \theta_3^{b,1} = 32.56\}$$

However, it follows that given the trading opportunities $\Psi^1(\omega_{51})$, the region $(\theta_3^{s,1}(\omega_{51}), \theta_4^{s,1}(\omega_{51})] = (31.22, 30.84]$ is not well defined, and submitting a limit sell order at price p_4 in state ω_{51} is dominated by submitting a limit sell order at p_3 or a limit buy order at p_1. In such cases, where the algorithm finds dominated order choices, it iteratively eliminates them from consideration until the resulting final sequence of indifference valuations is monotone. In this case, for instance, the algorithm determines the private valuation u^* that makes the trader indifferent between submitting a limit sell order at price p_3 and a limit buy order at price p_1.

$$u^* = \frac{\Psi_1^{b,1}(\omega_{51})p_1 + \Psi_3^{s,1}(\omega_{51})p_3}{\Psi_1^{b,1}(\omega_{51}) + \Psi_3^{s,1}(\omega_{51})} = \frac{0.18 \times 30 + 0.16 \times 32}{0.18 + 0.16} = 30.94. \tag{23}$$

Setting $\theta_3^{s,1} = \theta_4^{s,1} = \theta_1^{b,1} = u^* = 30.94$, the algorithm determines the optimal trading strategy[15]

$$\theta^1(\Psi^1(\omega_{51})) = \{29.08, 30.50, 30.94, 30.94, 30.94, 32.00, 32.56\}. \tag{24}$$

After finding the optimal trading strategy $\theta^1(\Psi^1(s_t))$ for all states $s_t \in \Omega$ in this fashion, the first iteration of the algorithm is completed. These iterations continue until we obtain convergence for the optimal strategies $\theta^n(s_t)$ and conditional execution probabilities $\Psi^n(s_t)$ respectively. The convergence limits $\theta^*(s_t)$ and $\Psi^*(s_t)$ for all $s_t \in \Omega$ characterize the symmetric stationary equilibrium of our dynamic limit order trading game. In our specific example, the conditional execution probabilities and optimal strategies for $s_t = \omega_{51}$ converge to

$$\Psi^{s,*}(\omega_{51}) = \{1.00, 0.8138, 0.2025, 0\}, \qquad \Psi^{b,*}(\omega_{51}) = \{0, 0.4618, 1.00, 0\},$$

$$\theta^*(\omega_{51}) = \{29, 30.6687, 31.3049, 31.3049, 31.3049, 31.3049, 32.8580\}. \tag{25}$$

Thus, we can characterize the optimal trading strategy $\theta^*(\omega_{51})$ in state $\omega_{51} = \{+2, 0, -1, 0\}$ as follows. It is never optimal for a trader to submit sell orders at prices $p_1 = 30$ and $p_4 = 33$ and buy orders at price $p_1 = 30$.[16] If the trader's private value u is in the interval [29, 30.6687], she will submit a limit sell order at $p_2 = 31$ with an execution probability of 0.8138. If $u \in (30.6687, 31.3049]$, she will submit a limit sell order at $p_3 = 32$ with an execution probability of 0.2025.[17] If $u \in (31.3049, 32.8580]$, she will submit a limit buy order at $p_2 = 31$ with an execution probability of 0.4618. Finally, if $u \in (32.8580, 34]$, she will submit a market buy order at $p_3 = 32$.

[15] Note that although placing a limit sell order at p_4 is dominated, we still set $\theta_4^{s,1} = 30.94$ for notational and computational convenience and it does not affect the following iterations of the algorithm.

[16] Consider why it is never optimal to submit a market sell order at $p_1 = 30$. Note that since the support of the probability distribution for private value u is [29, 34], the largest net payoff from selling at $p_1 = 30$ will be realized by a trader with the minimum possible private value $u = 29$, which is equal to $p_1 - u = 1$. However, since the conditional execution probability of a limit sell order placed at $p_2 = 31$ is substantial, i.e., $\Psi_2^{s,*} = 0.8138$, the trader with $u = 29$ will realize a larger expected payoff of $0.8138 \times (31 - 29) = 1.6276$ from placing a limit sell order at p_2. Therefore, she will prefer to submit a limit sell order at p_2 rather than submit a market sell order at p_1. The same comparison also applies to all traders with $u \in [29, 30]$. However, we can show that, ceteris paribus, if the trader's private value distribution has a wider support with [27, 36], submitting a market sell order at p_1 in state ω_{51} becomes optimal for all traders with $u \in [27, 28.0814]$.

[17] Note that in state ω_{51}, the conditional execution probability (0.8138) of a limit sell order placed at $p_2 = 31$ is much higher than that (0.2025) of a limit sell order placed at $p_3 = 32$. First, a sell order p_2 has price priority over a sell order at p_3. Second, due to time priority, a limit sell order placed at p_3 in state ω_{51} can only be executed after the execution or expiration of the existing limit sell order (with age 1) at p_3. Hence, the range of private values, [29, 30.6687], in which it is optimal to submit a limit sell order at p_2 is substantially wider than the range of private values, (30.6687, 31.3049], in which it is optimal to submit a limit sell order at p_3.

5. Results of the comparative dynamics analysis

We apply the algorithm explained in the previous section to solve for the equilibrium of our limit order market under various initial conditions and parameter specifications. Our objective is to address the following set of questions in our comparative dynamics analysis of the equilibrium limit order book. Does the numerical algorithm converge to a unique equilibrium given the underlying probability distribution of private valuations and the price grid? What is the limiting probability distribution of the Markov chain that characterizes the equilibrium limit order book, and which states of the book are visited most frequently or less frequently in the steady state? What is the effect of varying the dispersion of the private value (liquidity demand) distribution of the traders on the steady state of the equilibrium limit order book? What is the average conditional bid–ask spread in equilibrium and how does it vary in response to changes in the dispersion of traders' liquidity demand? What is the probability that the limit order book is empty on at least one side? What are the relative frequencies of limit orders and market orders respectively in the equilibrium order flow, and what is the effect of a change in traders' propensity to demand or supply liquidity on the choice between limit and market orders? What is the effect of increasing the life span of a limit order from two ($M=2$) periods to three ($M=3$) periods on the steady state of the equilibrium limit order book and the average conditional bid–ask spread? We provide answers to these questions in this section.

We find that, for a given uniform probability distribution of private valuations with support $[A, B]$ and a given price grid $P=\{p_1, \ldots, p_4\}$, our algorithm converges to a unique set of equilibrium indifference valuations θ^* and conditional execution probabilities Ψ^*. It does not matter whether the initial decision rules are state-independent or state-dependent. We find that the algorithm converges to the same unique equilibrium regardless of any type of feasible initial decision rule.

The equilibrium limit order book is characterized by the steady state distribution of the Markov chain that is obtained from the transition probability matrix Π of the states $s_t \in \Omega$. After having determined the optimal trading strategy $\theta^*(s_t)$ in equilibrium and given the private value probability distribution F of an agent, it is straightforward to derive the conditional order submission probabilities as explained in Section 3.1 and therefore, the one-step transition probability matrix Π of the equilibrium Markov chain. From Π, we derive the limiting probability distribution of the states of the limit order book in equilibrium.

In order to analyze the properties of the equilibrium limit order book, we first specify the price grid P, which is equal to $\{30, 31, 32, 33\}$. Then, we vary the support $[A, B]$ of the uniform probability distribution of the private valuation of the trader, where $A = p_1 - w$ and $B = p_4 + w$. We run the algorithm each time we chose a different closed interval $[A, B]$ for the probability distribution of the agent's valuation by varying the dispersion parameter w. After obtaining the equilibrium indifference valuations θ^* and the conditional execution probabilities Ψ^*, we solve for the limiting distribution of the Markov chain Π and obtain some summary statistics that characterize the equilibrium limit order book.

5.1. Equilibrium frequencies of limit order book states

In the first model specification, which is reported in Table 1 ($w = 1$), the support of the private value distribution is [29, 34], and limit orders last for $M=2$ periods. First, we notice that 19 out of 61 states are visited with positive probability in equilibrium. The most frequently hit state of the limit order book is the empty book state ω_1, which is visited with probability 0.2309. When we check the state transition matrix, which shows all the feasible two-way transitions among 61 states, we note that there are 40 different states from which the limit order book can make a transition to the empty book state ω_1 in the next period.[18] For any other state, the number of neighboring states from which

[18] For example, consider all states where there is only one outstanding limit order on one side of the book. If in such a state, the existing limit order is matched with a market order in the next period, the order book will revert to state ω_1. Further, consider all states where the are two existing limit orders on the same side of the book and the younger limit order with age 1 has price precedence over the older limit order with age 2. If the younger limit order is matched with a market order coming from the opposite side, the order book will also revert to state ω_1, since the older limit order with age 2 expires at the end of the current period.

Table 1
Probability distribution of private valuation: uniform(29,34) with $w = 1$.

Price grid: $P = \{30, 31, 32, 33\}$; maximum life span of a limit order: $M = 2$

Rank	Code	Prob.	P_1	P_2	P_3	P_4	Prob. market buy	Prob. market sell
19 out of 61 states visited in equilibrium								
1	ω_1	0.2309	0	0	0	0		
2	ω_3	0.0821	0	+1	0	0		0.2284
3	ω_8	0.0821	0	0	−1	0	0.2284	
4	ω_4	0.0686	0	0	+1	0		0.5689
5	ω_7	0.0686	0	−1	0	0	0.5689	
6	ω_{20}	0.0472	0	0	(+1,+2)	0		0.5689
7	ω_{23}	0.0472	0	(−1, −2)	0	0	0.5689	
8	ω_{35}	0.0461	0	+1	−2	0	0.3492	0.2284
9	ω_{53}	0.0461	0	+2	−1	0	0.2284	0.3492
10	ω_{41}	0.0409	0	−1	−2	0	0.5689	
11	ω_{47}	0.0409	0	+2	+1	0		0.5689
12	ω_{11}	0.0323	0	+2	0	0		0.2846
13	ω_{16}	0.0323	0	0	−2	0	0.2846	
14	ω_{12}	0.0268	0	0	+2	0		0.5464
15	ω_{15}	0.0268	0	−2	0	0	0.5464	
16	ω_{29}	0.0204	0	+1	+2	0		0.5663
17	ω_{59}	0.0204	0	−2	−1	0	0.5663	
18	ω_{19}	0.0202	0	(+1,+2)	0	0		0.2284
19	ω_{24}	0.0202	0	0	(−1, −2)	0	0.2284	
States never visited in equilibrium (bottom 5)								
57	ω_{56}	0	−2	−1	0	0	0.7286	
58	ω_{57}	0	−2	0	−1	0	0.7103	
59	ω_{58}	0	−2	0	0	−1	0.6928	
60	ω_{60}	0	0	−2	0	−1	0.5464	
61	ω_{61}	0	0	0	−2	−1	0.2846	
							Conditional averages	
							0.3851	0.3851

Average bid–ask spread 1

Number of hit states (both sides of the book nonempty) 2

Probability that the book is nonempty on both sides 9.22%

Buy P_1	Buy P_2	Buy P_3	Buy P_4	Sell P_1	Sell P_2	Sell P_3	Sell P_4	Total
Average conditional execution probabilities of limit orders								
0.0000	0.3060	0.7299	0.8816	0.8816	0.7299	0.3060	0.0000	
Percentages of submitted orders								
0.00%	28.41%	21.59%	0.00%	0.00%	21.59%	28.41%	0.00%	100.00%
Limit buy orders:			32.55%	**Limit sell orders:**			32.55%	
Market buy orders:			17.45%	**Market sell orders:**			17.45%	

the order book can make a transition to that specific state varies between 1 and 8. Thus, the empty book state ω_1 is a natural starting point, and it is not surprising that it is the most often visited state in equilibrium.

When we look at the other 18 states visited in equilibrium and their frequency rankings in Table 1, we note that traders prefer to submit their buy or sell orders either at p_2 or at p_3. In fact, no orders are submitted in equilibrium at prices $p_1 = 30$ and $p_4 = 33$ (see also "Percentage of Submitted Orders" in Table 1). This can be explained by the fact that in this particular specification, the support of the private value distribution, [29, 34], is very tightly placed around the price bin P. Therefore, in all states of the limit order book, submitting a buy order at p_4 is dominated by submitting a buy order at p_3, even though the conditional execution probability of a buy order submitted at p_4 is always greater than that of a buy order submitted at p_3. Similarly, submitting a sell order at p_1 is always dominated by submitting a sell order at p_2.

Without loss of generality, we can explain the above result by taking a closer look to the optimal order placement strategies $\theta^*(\omega_1)$ in state ω_1, i.e., the empty limit order book. Note that equilibrium trading opportunities in ω_1 are given by $\Psi^{b,*}(\omega_1) = \{0, 0.366, 0.814, 0.926\}$ and $\Psi^{s,*}(\omega_1) = \{0.926, 0.814, 0.366, 0\}$ on the buy and sell sides respectively. Then, we find that the optimal trading strategy is given by

$$\theta^*(\omega_1) = \{29, 30.183, 31.5, 31.5, 31.5, 31.5, 32.817, 34\}.$$

Thus, traders with private values in [29, 30.183] submit limit sell orders at $p_2 = 31$, which accounts for 23.66% of all submitted orders in state ω_1. Traders with private values in (30.183, 31.5] submit limit sell orders at $p_3 = 32$, accounting for another 23.66%. Traders with private values in (31.5, 32.817] submit limit buy orders at $p_2 = 31$ (26.34%). Finally, traders with private values in (32.817, 34] submit limit buy orders at $p_3 = 32$, accounting for another 23.66% of all orders submitted in ω_1.[19] The private value at which a trader is indifferent between submitting a sell order at $p_1 = 30$ or at $p_2 = 31$ is equal to 22.73, which is less than $A = 29$. Similarly, the private value at which a trader is indifferent between submitting a buy order at $p_3 = 32$ or at $p_4 = 33$ is equal to 40.27, which is greater than $B = 34$.[20] Since these indifference values are out of the range of the private value support [29, 34], limit buy order submissions at p_4 and limit sell order submissions at p_1 are dominated. In other words, in this range of private values, the trade-off between potential trading gains (from a lower bid price or a higher ask price relative to the trader's private value) and execution probability tilts in favor of potential trading gains.

Since it is never optimal for traders to submit a sell order at p_1 or a buy order at p_4 (i.e., relatively aggressive order strategies are not optimal) in the equilibrium shown in Table 1, it is also never optimal to submit a limit buy order at p_1 or a limit sell order at p_4 on the other side of the book respectively (these latter limit orders have zero execution probabilities). Hence, the most conservative limit order strategies are also dominated in the equilibrium shown in Table 1.

When we further look at the equilibrium frequency rankings of different states in Table 1, we notice that the empty book state ω_1 is followed by $\omega_3 = \{0, +1, 0, 0\}$, $\omega_8 = \{0, 0, -1, 0\}$, $\omega_4 = \{0, 0, +1, 0\}$, and $\omega_7 = \{0, -1, 0, 0\}$. One of these states will be visited right after ω_1 (see footnote 19). Suppose the limit order book is in state ω_3. Then, we find that

$$\Psi^{b,*}(\omega_3) = \{0, 0.203, 0.814, 0.926\}, \qquad \Psi^{s,*}(\omega_3) = \{0, 1, 0.462, 0\},$$

$$\theta^*(\omega_3) = \{0, 30.142, 31.695, 31.695, 31.695, 31.695, 32.331, 34\}$$

in equilibrium.[21] Note that in this state, a limit buy order submitted at $p_2 = 31$ is much less likely to be executed than a limit buy order submitted at $p_3 = 32$ (0.203 vs. 0.814 respectively), since there is already an existing limit buy order at p_2, which has time priority. Hence, if a trader has a sufficiently high private value for the asset, she will submit a limit buy order at p_3 in order to gain price priority over the existing limit buy order at p_2. Therefore, in state $\omega_3 = \{0, +1, 0, 0\}$, we observe that the percentage of buy order submission at p_3, 33.37%, is much greater than that at p_2, 12.72%. Consistent with this finding, we note in Table 1 that state $\omega_{47} = \{0, +2, +1, 0\}$ is more than twice likely to be visited than

[19] These order submission percentages in state ω_1 also help us explain why states $\omega_3 = \{0, +1, 0, 0\}$ and $\omega_8 = \{0, 0, -1, 0\}$ are slightly more likely to be observed than the states $\omega_4 = \{0, 0, +1, 0\}$ and $\omega_7 = \{0, -1, 0, 0\}$ respectively as shown in Table 1.

[20] We solve the following two equations to find these indifference values: $0.926(30 - u) = 0.814(31 - u)$ and $0.926(u - 33) = 0.814(u - 32)$.

[21] Thus, traders with private values in [29, 30.142] submit market sell orders at $p_2 = 31$, 22.84% of all submitted orders in state ω_3. Traders with private values in (30.142, 31.695] submit limit sell orders at $p_3 = 32$, accounting for another 31.06%. Traders with private values in (31.695, 32.331] submit limit buy orders at $p_2 = 31$, accounting for 12.72%. Finally, traders with private values in (32.331, 34] submit limit buy orders at $p_3 = 32$, accounting for 33.37%.

state $\omega_{19} = \{0, (+1, +2), 0, 0\}$.[22] Symmetric arguments also apply to state $\omega_8 = \{0, 0, -1, 0\}$ as a starting point, and help us explain the greater relative likelihood of state $\omega_{41} = \{0, -1, -2, 0\}$ over $\omega_{24} = \{0, 0, (-1, -2), 0\}$ in the equilibrium shown in Table 1.

One can also note in state $\omega_3 = \{0, +1, 0, 0\}$ that although the execution probability of a limit sell order at p_3 is relatively low compared to that of a market sell order at p_2 (0.462 vs. 1 respectively), there is still a substantial mass of patient sellers (accounting for 31.06% of all submitted orders in ω_3), who are willing to provide liquidity by placing limit sell orders at price p_3 in exchange for a higher potential trading gain in future periods. Therefore, state $\omega_{53} = \{0, +2, -1, 0\}$, which follows ω_3 after a limit sell order submission at p_3, is also one of the more likely observed states in Table 1.[23] Symmetric arguments also apply to state $\omega_{35} = \{0, +1, -2, 0\}$, which follows state $\omega_8 = \{0, 0, -1, 0\}$ after a limit buy order submission at p_2.

Next, let us consider state $\omega_4 = \{0, 0, +1, 0\}$, which is also one of the most frequently visited states in the equilibrium shown in Table 1. In this state, we find that

$$\Psi^{b,*}(\omega_4) = \{0, 0.184, 0.556, 0.926\}, \qquad \Psi^{s,*}(\omega_4) = \{0, 0, 1, 0\},$$

$$\theta^*(\omega_4) = \{0, 0, 31.845, 31.845, 31.845, 31.845, 32.495, 34\}$$

in equilibrium.[24] One should note that in this state, limit buy order submissions at $p_3 = 32$ are much more likely than those at $p_2 = 31$ (order submission percentages are 30.10% and 13.01% respectively), since due to price priority, the execution probability of a limit buy order at p_2, 0.184, is much smaller than that of a limit buy order at p_3, 0.556. Therefore, state $\omega_{20} = \{0, 0, (+1, +2), 0\}$ is much more likely to be observed than state $\omega_{29} = \{0, +1, +2, 0\}$ in equilibrium, as shown in Table 1.[25,26] Symmetric arguments also apply to state $\omega_7 = \{0, -1, 0, 0\}$ as a starting point, and help us explain the greater relative likelihood of state $\omega_{23} = \{0, (-1, -2), 0, 0\}$ over $\omega_{59} = \{0, -2, -1, 0\}$ in the equilibrium shown in Table 1.

Finally, we notice that in the equilibrium shown in Table 1, state $\omega_{11} = \{0, +2, 0, 0\}$ is observed after market orders hit the limit order book in states $\omega_{35} = \{0, +1, -2, 0\}$ (market buy order at p_3), $\omega_{19} = \{0, (+1, +2), 0, 0\}$, and $\omega_{29} = \{0, +1, +2, 0\}$, which we discussed above. Similarly, state $\omega_{12} = \{0, 0, +2, 0\}$ is observed after a market sell order at p_3 hits the limit order book in state $\omega_{20} = \{0, 0, (+1, +2), 0\}$.[27]

In subsequent model specifications, we notice that as the dispersion in traders' private valuations increases (i.e., the parameter w increases and the private value support $[A, B]$ widens), the percentage of orders submitted at prices p_1 and p_4 gradually increases and one can observe states with depths at prices p_1 and p_4 more frequently in the equilibrium limit order book. Corresponding to this regularity, we see that the overall number of states ever visited in the equilibrium increases, reaching 53 in the model specification shown in Table 2 ($w = 4$), where the support of the private value distribution is [26, 37].

As w increases, we also find that the equilibrium frequency rankings of certain states are affected. For example, submitting a limit buy order at $p_4 = 33$ becomes optimal for an increasingly larger mass of traders (with larger private values) in state $\omega_4 = \{0, 0, +1, 0\}$ as w increases. Similarly, submitting a limit sell order at $p_1 = 30$ becomes optimal for an increasingly larger mass of traders (with smaller

22 If a trader submits a limit buy order at p_3 (p_2) in state ω_3, the next state will be ω_{47} (ω_{19}). Note also that transitions to state ω_{47} are also possible from states ω_{19} itself and $\omega_{29} = \{0, +1, +2, 0\}$, which are observed with positive probability in the equilibrium shown in Table 1. Similarly, transitions to state ω_{19} are also possible from ω_{19} itself, ω_{29}, and $\omega_{35} = \{0, +1, -2, 0\}$.

23 Note that transitions to ω_{53} can also follow ω_{19} and ω_{35}, which are observed with positive probability in the equilibrium shown in Table 1.

24 Thus, traders with private values in [29, 31.845] submit market sell orders at $p_3 = 32$ (56.89% of all submitted orders in ω_4). Traders with private values in (31.845, 32.495] submit limit buy orders at $p_2 = 31$ (13.01% of all orders in ω_4). Finally, traders with private values in (32.495, 34] submit limit buy orders at $p_3 = 32$ (30.10% of all orders in ω_4).

25 Note that if a trader submits a limit buy order at p_3 (p_2) in state ω_3, the next state will be ω_{20} (ω_{29}).

26 Transitions to state ω_{20} can also occur from states ω_{20} itself and ω_{47}, which are observed with positive probability in the equilibrium shown in Table 1.

27 By symmetry, state $\omega_{16} = \{0, 0, -2, 0\}$ is observed after market orders hit the limit order book in states $\omega_{53} = \{0, +2, -1, 0\}$ (market sell order at p_2), $\omega_{24} = \{0, 0, (-1, -2), 0\}$, and $\omega_{59} = \{0, -2, -1, 0\}$, which we discussed above. State $\omega_{15} = \{0, -2, 0, 0\}$ is observed after a market buy order at p_2 hits the limit order book in state $\omega_{23} = \{0, (-1, -2), 0, 0\}$.

Table 2
Probability distribution of private valuation: uniform (26,37) with $w = 4$.

Price grid: $P = \{30, 31, 32, 33\}$; maximum life span of a limit order: $M = 2$

Rank	Code	Prob.	P_1	P_2	P_3	P_4	Prob. market buy	Prob. market sell
Most frequently visited states in equilibrium								
1	ω_1	0.2674	0	0	0	0		
2	ω_3	0.0798	0	+1	0	0		0.3243
3	ω_8	0.0798	0	0	−1	0	0.3243	
4	ω_5	0.0395	0	0	0	+1		0.6021
5	ω_6	0.0395	−1	0	0	0	0.6021	
6	ω_4	0.0316	0	0	+1	0		0.5071
7	ω_7	0.0316	0	−1	0	0	0.5071	
8	ω_{41}	0.0302	0	−1	−2	0	0.5071	
9	ω_{47}	0.0302	0	+2	+1	0		0.5071
10	ω_{21}	0.0208	0	0	0	(+1,+2)		0.6021
11	ω_{22}	0.0208	(−1, −2)	0	0	0	0.6021	
12	ω_{38}	0.0208	−1	−2	0	0	0.6021	
13	ω_{49}	0.0208	0	0	+2	+1		0.6021
14	ω_{11}	0.0206	0	+2	0	0		0.3848
15	ω_{16}	0.0206	0	0	−2	0	0.3848	
16	ω_{35}	0.0154	0	+1	−2	0	0.4267	0.3243
17	ω_{53}	0.0154	0	+2	−1	0	0.3243	0.4267
18	ω_{39}	0.0138	−1	0	−2	0	0.6021	
19	ω_{48}	0.0138	0	+2	0	+1		0.6021
20	ω_{13}	0.0125	0	0	0	+2		0.5813
21	ω_{14}	0.0125	−2	0	0	0	0.5813	
22	ω_{29}	0.0103	0	+1	+2	0		0.5109
23	ω_{59}	0.0103	0	−2	−1	0	0.5109	
24	ω_{30}	0.0099	0	+1	0	+2		0.6018
25	ω_{57}	0.0099	−2	0	−1	0	0.6018	
26	ω_{33}	0.0099	+1	0	−2	0	0.3848	0.0747
27	ω_{54}	0.0099	0	+2	0	−1	0.0747	0.3848
28	ω_2	0.0080	+1	0	0	0		0.0747
29	ω_9	0.0080	0	0	0	−1	0.0747	
Least frequently visited states in equilibrium (bottom 10)								
44	ω_{43}	0.0032	0	0	−1	−2	0.3243	
45	ω_{44}	0.0032	+2	+1	0	0		0.3243
46	ω_{34}	0.0010	1	0	0	−2	0.2224	0.0747
47	ω_{52}	0.0010	+2	0	0	−1	0.0747	0.2224
48	ω_{40}	0.0010	−1	0	0	−2	0.6021	
49	ω_{46}	0.0010	+2	0	0	+1		0.6021
50	ω_{42}	0.0008	0	−1	0	−2	0.5071	
51	ω_{45}	0.0008	+2	0	+1	0		0.5071
52	ω_{20}	0.0004	0	0	(+1,+2)	0		0.5071
53	ω_{23}	0.0004	0	(−1, −2)	0	0	0.5071	
States never visited in equilibrium								
54	ω_{18}	0	(+1,+2)	0	0	0		0.0747
55	ω_{25}	0	0	0	0	(−1, −2)	0.0747	
56	ω_{26}	0	+1	+2	0	0		0.3561
57	ω_{27}	0	+1	0	+2	0		0.5169
58	ω_{28}	0	+1	0	0	+2		0.5813
59	ω_{58}	0	−2	0	0	−1	0.5813	
60	ω_{60}	0	0	−2	0	−1	0.5169	
61	ω_{61}	0	0	0	−2	−1	0.3561	
							Conditional averages	
							0.4026	0.4026

Average bid–ask spread
1.433

Number of hit states (both sides of the book nonempty)
12

Table 2 (*Continued*)

Price grid: P = {30, 31, 32, 33}; maximum life span of a limit order: M = 2								
Rank	Code	Prob.	P_1	P_2	P_3	P_4	Prob. market buy	Prob. market sell
Probability that the book is nonempty on both sides								
7.68%								

Buy				Sell				Total
P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4	
Average conditional execution probabilities of limit orders								
0.1831	0.3795	0.5515	0.7913	0.7913	0.5515	0.3795	0.1831	
Percentages of submitted orders								
9.78%	17.28%	12.80%	10.14%	10.14%	12.80%	17.28%	9.78%	100.00%
Limit buy orders:			32.15%	**Limit sell orders:**			32.15%	
Market buy orders:			17.85%	**Market sell orders:**			17.85%	

private values) in state $\omega_7 = \{0, -1, 0, 0\}$.[28] Consequently, in the equilibria of specifications with $w > 1$, we find that states $\omega_{49} = \{0, 0, +2, +1\}$ and $\omega_{38} = \{-1, -2, 0, 0\}$ are visited more frequently (their frequency ranks gradually increase), whereas states $\omega_{20} = \{0, 0, (+1, +2), 0\}$ and $\omega_{23} = \{0, (-1, -2), 0, 0\}$ are visited gradually less often in equilibrium as w increases.[29] Consistent with our earlier intuition that the rules of price priority and time priority are important determinants of equilibrium state frequencies, we also find that states $\omega_{49} = \{0, 0, +2, +1\}$ and $\omega_{38} = \{-1, -2, 0, 0\}$ are much more likely to be visited than states $\omega_{29} = \{0, +1, +2, 0\}$ and $\omega_{59} = \{0, -2, -1, 0\}$ respectively, as shown in the equilibrium of the model in Table 2.

As shown in Table 2, we find that states $\omega_5 = \{0, 0, 0, +1\}$, $\omega_6 = \{-1, 0, 0, 0\}$, $\omega_2 = \{+1, 0, 0, 0\}$, and $\omega_9 = \{0, 0, 0, -1\}$ are in the list of states visited with positive probability in equilibrium when $w = 4$, even though they are visited with zero limiting probability in equilibrium when $w \leq 3$. When we look at the optimal order strategies $\theta^*(\omega_1)$ of a trader arriving at the empty order book state ω_1 when $w = 4$, we find that submitting a limit buy order at p_4 and submitting a limit sell order at p_1 become optimal strategies for some very impatient traders for the very first time

$$\Psi^{b,*}(\omega_1) = \{0.190, 0.434, 0.608, 0.842\}, \qquad \Psi^{s,*}(\omega_1) = \{0.842, 0.608, 0.434, 0.190\},$$

$$\theta^*(\omega_1) = \{27.393, 28.505, 31.225, 31.5, 31.5, 31.775, 34.495, 35.607\}.$$

in equilibrium.[30] In other words, these two strategies are dominated for any trader arriving in state ω_1 in all specifications with $w \leq 3$, but not so when $w \geq 4$. Thus, when $[A, B] = [26, 37]$, there exist some traders with extremely high (low) private values who demand immediacy by submitting buy (sell) limit orders at p_4 (p_1), for whom the trade-off between execution probability and the size of potential trading gain tilts in favor of execution probability. Consequently, we observe that states $\omega_5 = \{0, 0, 0, +1\}$ and $\omega_6 = \{-1, 0, 0, 0\}$ are among often visited states in the equilibrium shown in Table 2. At the same time, on the opposite sides of the order book, we find that it becomes optimal for some relatively small mass of traders with moderate private values to submit a limit buy order at p_1 or a limit sell order at p_4 in state ω_1 to match the liquidity demands of traders with extreme private values. Hence, these most conservative limit order strategies are not dominated anymore in the equilibrium shown

[28] These order strategies first become optimal for some traders in these states when $w = 2$.

[29] Note that in the model specification shown in Table 2, ω_{20} and ω_{23} are the least often visited states in equilibrium.

[30] Thus, if $u \in [26, 27.393]$, traders submit limit sell orders at p_1 (12.66% of all submitted orders). If $u \in (27.393, 28.505]$, they submit limit sell orders at p_2 (10.11%). If $u \in (28.505, 31.225]$, they submit limit sell orders at p_3 (24.72%). If $u \in (31.225, 31.5]$, they submit limit sell orders at p_4 (2.50%). If $u \in (31.5, 31.775]$, they submit limit buy orders at p_1 (2.50%). If $u \in (31.775, 34.495]$, they submit limit buy orders at p_2 (24.72%). If $u \in (34.495, 35.607]$, they submit limit buy orders at p_3 (10.11%). Finally, if $u \in (35.607, 37]$, they submit limit buy orders at p_4 (12.66%).

in Table 2, and we observe that states $\omega_2 = \{+1, 0, 0, 0\}$, and $\omega_9 = \{0, 0, 0, -1\}$ are visited with positive probability in equilibrium.

As a general rule, we note that when we compare two states, where there are two limit orders at the same side of the book in each state, we observe that the state where the limit order with age 1 outbids the other limit order with age 2 is always observed more often in equilibrium than the state where the limit order with age 2 outbids the other limit order with age 1. Thus, due to time and price priority rules, the depth of the quotes at a given state of the limit order book is important in determining the equilibrium optimal trading strategy of the arriving trader. For example, if the current state of the market is $\omega_8 = \{0, 0, -1, 0\}$, which is among the most frequently observed equilibrium states in all specifications, then a potential seller is less likely to submit a sell order at p_4 or p_3 according to our results. This is intuitive because of both the time priority and the price priority of the existing limit order at p_3. Hence, we observe that state $\omega_{41} = \{0, -1, -2, 0\}$ is much more likely to be visited in equilibrium than both $\omega_{24} = \{0, 0, (-1, -2), 0\}$ and $\omega_{61} = \{0, 0, -2, -1\}$ as shown in the equilibria of Tables 1 and 2. Similarly, state $\omega_{47} = \{0, +2, +1, 0\}$ that follows $\omega_3 = \{0, +1, 0, 0\}$ is much more likely to be visited in equilibrium than both $\omega_{19} = \{0, (+1, +2), 0, 0\}$ and $\omega_{26} = \{+1, +2, 0, 0\}$ as shown in Tables 1 and 2 as well.

Finally, when we look at the list of never visited states in the stationary equilibrium of the model (when $w \leq 4$), we observe the following states:

$$\omega_{18} = \{(+1, +2), 0, 0\}, \qquad \omega_{25} = \{0, 0, 0, (-1, -2)\},$$
$$\omega_{26} = \{+1, +2, 0, 0\}, \qquad \omega_{27} = \{+1, 0, +2, 0\}, \qquad \omega_{28} = \{+1, 0, 0, +2\},$$
$$\omega_{60} = \{0, -2, 0, -1\}, \qquad \omega_{61} = \{0, 0, -2, -1\}, \qquad \omega_{58} = \{-2, 0, 0, -1\}.$$

One reason why these states are never observed in equilibrium (for $w \leq 4$) is that conditional execution probabilities of limit buy orders placed at p_1 or limit sell orders placed at p_4 are very small in many states when $w \leq 4$. In fact, as we mentioned above, it is optimal for some traders to submit a limit buy order at p_1 or a limit sell order at p_4 in certain states (e.g., ω_1) only if $w \geq 4$. Further, submitting a limit buy order at p_1 in state ω_2 and submitting a limit sell order at p_4 in state ω_9 are still dominated strategies when $w = 4$, so that states ω_{18} and ω_{25} have zero limiting probabilities in equilibrium as shown in Table 2. In addition, for some of the above states, we note that limit orders with age 1 are lagging behind limit orders with age 2 in terms of both price priority and time priority rules, which is the second reason why these states are never observed in equilibrium when $w \leq 4$. On the other hand, Table 2 shows that when $w = 4$, the following states are observed with small, but albeit positive probabilities in equilibrium:

$$\omega_{44} = \{+2, +1, 0, 0\}, \qquad \omega_{45} = \{+2, 0, +1, 0\}, \qquad \omega_{46} = \{+2, 0, 0, +1\},$$
$$\omega_{42} = \{0, -1, 0, -2\}, \qquad \omega_{43} = \{0, 0, -1, -2\}, \qquad \omega_{40} = \{-1, 0, 0, -2\}.$$

Note that in these states of the order book, younger limit orders outbid older limit orders at the same side of the book, which is consistent with expected-utility-maximizing liquidity traders facing price priority and time priority rules.

5.2. Bid–ask spreads

One of the most important statistics that characterizes liquidity provision in a limit order market is the average bid–ask spread. A common regularity that we observe in our numerical results for different specifications of our model is that the average bid–ask spread (conditional on both sides of the book being nonempty) is monotonically increasing in the dispersion w of traders' private value distribution.

In our model specifications with $N = 4$ and $M = 2$, there are only 12 states in which both sides of the book are nonempty. In the model specification of Table 1 with $w = 1$, only two such states are visited in equilibrium: ω_{35} and ω_{53}. In this case, the average bid–ask spread is exactly equal to 1 since order submissions at p_1 or at p_4 are dominated as we explained above when $w = 1$. Further, the probability that the book is nonempty on both sides is equal to 0.0922 in this specification. However, as the private value dispersion parameter w increases, we find that the number of nonempty two-sided states visited in equilibrium increases as well, corresponding to the increase in the overall number of states observed in equilibrium. In specifications with $w \geq 2$, we find that the percentage of buy (sell) orders submitted at price p_4 (p_1) in equilibrium is positive, and this percentage increases with the dispersion w in private values. In other words, as the dispersion in traders' private valuations increases, there starts to exist a larger mass of impatient traders (with larger liquidity demands) on both tails of the private value distribution, and these traders submit more aggressive limit orders or market orders at more extreme prices in order to obtain larger execution probabilities in equilibrium, thereby consuming more liquidity. On the other hand, this increase in liquidity demand is matched by the liquidity supply of traders with more moderate private values (close to the mean), who start to find submitting more conservative limit orders (i.e., limit buy orders at p_1 and limit sell orders at p_4) more profitable as w increases, since the conditional execution probabilities of such limit orders increase in equilibrium as well (see Tables 1 and 2).[31]

Consistent with the above economic intuition, we find that in the equilibrium shown in Table 2 with $w = 4$, all 12 states in which both sides of the book are nonempty are visited with positive probability, and the probability-weighted average bid–ask spread is equal to 1.433. The probability that the book is nonempty on both sides is equal to 0.0768. In general, we find that the average bid–ask spread is increasing in the dispersion in traders' private valuations for the asset. If $w = 2$, the average bid–ask spread is still equal to 1 as ω_{35} and ω_{53} are still the only two equilibrium states with both sides being nonempty. In this specification, the overall number of states visited in equilibrium increases to 25. Therefore, the probability that the book is nonempty on both sides is equal to 0.0651. If $w = 3$, the probability-weighted average bid–ask spread increases to 1.0657, as the number of equilibrium states with both sides being nonempty increases to 10. In this specification, the overall number of states visited in equilibrium increases to 41, and the probability that the book is nonempty on both sides is equal to 0.0577.[32]

Our findings regarding the average bid–ask spread are also consistent with our results on the steady state distribution of the limit order book, which we analyzed in the previous subsection. In our model specifications with $w \leq 4$, the second and third most frequently visited states are $\omega_3 = \{0, (+1, 1), 0, 0\}$ and $\omega_8 = \{0, 0, (-1, 1), 0\}$, whereas if $w \geq 5$, those two states are $\omega_5 = \{0, 0, 0, (+1, 1)\}$ and $\omega_6 = \{(-1, 1), 0, 0, 0\}$, respectively. This result is consistent with traders with extreme private values submitting even more aggressively priced orders as the dispersion in traders' private valuations further increases ($w \geq 5$). Similarly, we find that for $w \geq 5$, conditional execution probabilities of buy (sell) limit orders at p_1 (p_4) continue to be increasing with the dispersion parameter w as well, and therefore, traders with moderate private values (close to the mean) tend to gradually submit more conservative limit orders at the corners of the price grid P.[33]

To further investigate the properties of the dynamic equilibrium of the limit order market in our model, and the average bid–ask spreads in particular, we allow limit orders to last for three periods, i.e., we let $M = 3$ and numerically solve for the equilibrium of our model where the price grid is still $P = \{30, 31, 32, 33\}$ as before. From Eq. (4), it follows that if $N = 4$ and $M = 3$, the total number of states is equal to $S = 369$. Table 3 shows the equilibrium of our model when $M = 3$ and $w = 1$. The number of states visited in equilibrium with a positive probability is 65. In 12 of these states, both sides of the book are nonempty with a probability of 0.1636. Since order submissions at p_1 or p_4 are also dominated in all states in this specification with $w = 1$, the average conditional bid–ask spread is exactly equal to 1. As the dispersion parameter w increases, buy (sell) order submissions at p_4 (p_1) start to become

[31] Note that as the dispersion parameter w of traders' private valuations increases from 1 in Table 1 to 4 in Table 2, the size of potential trading gains (for traders with extreme private values) increases substantially with respect to the width of the price grid P and therefore, relative to the tick size. Thus, for some liquidity-demanding traders with extreme private values, the trade-off between a less advantageous order price and higher execution probability clearly tilts in favor of execution probability.

[32] For $w \geq 5$, we also find that the average bid–ask spread monotonically increases with the dispersion parameter w. If $w = 5$ ($w = 6$), the probability-weighted average bid–ask spread is 1.7880 (2.1279), the number of states visited with positive probability in equilibrium is 57 (49), and the probability that both sides of the book being nonempty is equal to 0.0747 (0.0661).

[33] Note that if the price grid P were not fixed, we would have possibly observed sell (buy) order submissions at prices less than p_1 (more than p_4). But since this is not feasible by construction, we observe an overaccumulation of order submissions at the corners of the price grid when $w \geq 5$, and the conditional execution probabilities of buy (sell) limit orders submitted at p_1 (p_4) increase substantially when w increases beyond 4.

Table 3
Probability distribution of private valuation: uniform (29,34) with $w = 1$.

Price grid: $P = \{30, 31, 32, 33\}$; maximum life span of a limit order: $M = 3$

Rank	Code	Prob.	P_1	P_2	P_3	P_4	Prob. market buy	Prob. market sell
Most frequently visited states in equilibrium								
1	ω_1	0.1430	0	0	0	0		
2	ω_3	0.0563	0	+1	0	0		0.1421
3	ω_8	0.0563	0	0	−1	0	0.1421	
4	ω_4	0.0401	0	0	+1	0		0.5662
5	ω_7	0.0401	0	−1	0	0	0.5662	
6	ω_{11}	0.0321	0	+2	0	0		0.2022
7	ω_{16}	0.0321	0	0	−2	0	0.2022	
8	ω_{67}	0.0313	0	+1	−2	0	0.3351	0.2616
9	ω_{121}	0.0313	0	+2	−1	0	0.2616	0.3351
10	ω_{19}	0.0299	0	+3	0	0		0.2487
11	ω_{24}	0.0299	0	0	−3	0	0.2487	
12	ω_{12}	0.0237	0	0	+2	0		0.5580
13	ω_{15}	0.0237	0	−2	0	0	0.5580	
14	ω_{97}	0.0233	0	−1	−2	0	0.5662	
15	ω_{115}	0.0233	0	+2	+1	0		0.5662
16	ω_{28}	0.0197	0	0	(+1,+2)	0		0.5815
17	ω_{31}	0.0197	0	(−1, −2)	0	0	0.5815	
18	ω_{52}	0.0166	0	0	(+1,+2,+3)	0		0.5815
19	ω_{55}	0.0166	0	(−1, −2, −3)	0	0	0.5815	
20	ω_{145}	0.0143	0	+2	(−1, −3)	0	0.2616	0.3351
21	ω_{247}	0.0143	0	(+1,+3)	−2	0	0.3351	0.2616
22	ω_{79}	0.0134	0	+1	−3	0	0.3404	0.1421
23	ω_{175}	0.0134	0	+3	−1	0	0.1421	0.3404
24	ω_{36}	0.0109	0	0	(+1,+3)	0		0.5662
25	ω_{39}	0.0109	0	(−1, −3)	0	0	0.5662	
26	ω_{103}	0.0106	0	−1	−3	0	0.5662	
27	ω_{169}	0.0106	0	+3	+1	0		0.5662
28	ω_{193}	0.0104	0	+3	(+1,+2)	0		0.5815
29	ω_{235}	0.0104	0	(−1, −2)	(−3)	0	0.5815	
30	ω_{27}	0.0101	0	(+1,+2)	0	0		0.1150
31	ω_{32}	0.0101	0	0	(−1, −2)	0	0.1150	
Least frequently visited states in equilibrium (bottom 6)								
60	ω_{51}	0.0023	0	(+1,+2,+3)	0	0		0.1150
61	ω_{56}	0.0023	0	0	(−1, −2, −3)	0	0.1150	
62	ω_{127}	0.0017	0	+2	+3	0		0.5497
63	ω_{211}	0.0017	0	−3	−2	0	0.5497	
64	ω_{217}	0.0011	0	−3	(−1, −2)	0	0.5650	
65	ω_{223}	0.0011	0	(+1,+2)	+3	0		0.5650
							Conditional averages	
							0.3726	0.3726

Average bid–ask spread	**Number of hit states (both sides of the book nonempty)**
1	12

Probability that the book is nonempty on both sides
16.36%

Buy				Sell				Total
P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4	
Average conditional execution probabilities of limit orders								
0.0000	0.3256	0.8225	0.9398	0.9398	0.8225	0.3256	0.0000	
Percentages of submitted orders								
0.00%	28.81%	21.19%	0.00%	0.00%	21.19%	28.81%	0.00%	100.00%
Limit buy orders:			31.33%	**Limit sell orders:**			31.33%	
Market buy orders:			18.67%	**Market sell orders:**			18.67%	

optimal for some traders with extreme private values in an increasing number of states as it was the case above when $M=2$. However, there is one important difference. Since limit orders last for $M=3$ periods, conditional execution probabilities of limit orders submitted at various prices are higher (or not lower) in many states compared to the earlier setting where limit orders last for only $M=2$ periods. Thus, the incentives of traders with more extreme private values to submit more aggressive orders still increases with the dispersion in private values, albeit at a slower rate compared to the case with $M=2$. This difference renders the limit order book more liquid and causes the average bid–ask spreads to be lower relative to the case with $M=2$.

In the equilibrium of the limit order book shown in Table 4 with $M=3$ and $w=4$, we find that 215 states are visited with positive probability, and in 70 of these states, both sides of the book are nonempty with a probability of 0.0955. The average conditional bid–ask spread is equal to 1.0699. In untabulated specifications with $M=3$, we find that if $w=2$ ($w=3$), the average conditional bid–ask spread is equal to 1 (1.0019), and the probability that both sides of the book are nonempty is equal to 0.1010 (0.0876).[34] Thus, our results show that average equilibrium bid–ask spreads in specifications with $M=3$ are lower than those in corresponding specifications with $M=2$. Similarly, for any value of the dispersion parameter w, the equilibrium probability of both sides of the book being nonempty is higher when $M=3$ than when $M=2$. These findings suggest that a limit order market becomes more liquid when limit orders have longer times until expiration. Further, we also observe in Tables 3 and 4 that when $M=3$, the equilibrium probability of visiting the empty limit order book (state ω_1) is considerably lower than the same probability when $M=2$ in Tables 1 and 2 respectively.

If we look at the equilibrium probabilities and frequency rankings of certain states in Table 4 and compare them to the corresponding ones in Table 2, we can better understand why average conditional bid–ask spreads in specifications with $M=3$ are lower than those in specifications with $M=2$. In Table 4, we notice that the equilibrium probability of both states $\omega_5=\{0,0,0,+1\}$ and $\omega_6=\{-1,0,0,0\}$ is 0.0058 each when $w=4$ and $M=3$, whereas Table 2 reports that the same states are visited with probability 0.0395 when $w=4$ and $M=2$. Similarly, states $\omega_2=\{+1,0,0,0\}$ and $\omega_9=\{0,0,0,-1\}$ are both visited with zero probability in the equilibrium shown in Table 4, whereas they are visited with a positive probability (0.008 each) in the equilibrium shown in Table 2.[35] This comparison shows that when limit orders last for three periods, the incentives of traders with extreme private values to submit more aggressive orders in equilibrium (as the dispersion of private values increases) increases at a slower rate compared to the earlier case when limit orders last for two periods. As the conditional execution probabilities of limit orders submitted at p_2 or p_3 are increasing with the time to expiration M (see the section "Average conditional execution probabilities of limit orders" in Tables 2 and 4), traders with extreme private values behave more patiently in equilibrium when M is higher (see also the section "Percentages of submitted orders" in Tables 2 and 4). Similarly, in the equilibrium shown in Table 4, traders with moderate private values are not as eager in placing conservative limit orders at the corners of the price grid as they are in the equilibrium shown in Table 2.

5.3. Limit orders vs. market orders

As we discussed in Lemma 1 and Proposition 1, our theoretical model restricts traders' optimal order submission strategies to be a monotone function of their private valuations. This monotonicity property, which was first explored by Sandås (2001) and Hollifield et al. (2004), implies that agents with more extreme private values will be more likely to submit market orders and those with moderate private values will be more likely to submit limit orders. This feature does also hold in our model, and the optimal order placement strategies outlined in Proposition 2 (see Eqs. (11)–(14)) specify that traders with extremely high private values submit market buy orders, $d_L^{*b}(s_t)=1$ if and only if

[34] For $w \geq 5$, we again find that the average bid–ask spread monotonically increases with the dispersion parameter w. If $w=5$ ($w=6$), the probability-weighted average bid–ask spread is 1.1857 (1.4147), the number of states visited with positive probability in equilibrium is 259 (263), and the probability that both sides of the book being nonempty is equal to 0.0893 (0.0856).

[35] If $M=3$ and $w=5$, states ω_2 and ω_9 are visited with positive probability (0.0044 each). In this case, states ω_5 and ω_6 are the fourth and fifth most often visited states with an equilibrium probability of 0.0308 each.

Table 4
Probability distribution of private valuation: uniform (26,37) with $w = 4$.

Price grid: $P = \{30, 31, 32, 33\}$; maximum life span of a limit order: $M = 3$

Rank	Code	Prob.	P_1	P_2	P_3	P_4	Prob. market buy	Prob. market sell
Most frequently visited states in equilibrium								
1	ω_1	0.1864	0	0	0	0		
2	ω_3	0.0669	0	+1	0	0		0.2707
3	ω_8	0.0669	0	0	−1	0	0.2707	
4	ω_4	0.0479	0	0	+1	0		0.5072
5	ω_7	0.0479	0	−1	0	0	0.5072	
6	ω_{97}	0.0274	0	−1	−2	0		0.5072
7	ω_{115}	0.0274	0	+2	+1	0	0.5072	
8	ω_{19}	0.0273	0	+3	0	0		0.3532
9	ω_{24}	0.0273	0	0	−3	0	0.3532	
10	ω_{11}	0.0234	0	+2	0	0		0.3035
11	ω_{16}	0.0234	0	0	−2	0	0.3035	
12	ω_{67}	0.0184	0	+1	−2	0	0.3603	0.4130
13	ω_{121}	0.0184	0	+2	−1	0	0.4130	0.3603
14	ω_{20}	0.0173	0	0	+3	0		0.5141
15	ω_{23}	0.0173	0	−3	0	0	0.5141	
16	ω_{94}	0.0155	−1	−2	0	0	0.5993	
17	ω_{117}	0.0155	0	0	+2	+1		0.5993
18	ω_{103}	0.0086	0	−1	−3	0	0.5662	
19	ω_{169}	0.0086	0	+3	+1	0		0.5662
20	ω_{12}	0.0086	0	0	+2	0		0.5233
21	ω_{15}	0.0086	0	−2	0	0	0.5233	
22	ω_{298}	0.0085	−1	−2	−3	0	0.5993	
23	ω_{353}	0.0085	0	+3	+2	+1		0.5993
24	ω_{13}	0.0078	0	0	0	+2		0.5915
25	ω_{14}	0.0078	−2	0	0	0	0.5915	
26	ω_{61}	0.0069	0	+1	+2	0		0.5188
27	ω_{151}	0.0069	0	−2	−1	0	0.5188	
28	ω_{195}	0.0066	0	0	+3	(+1,+2)		0.6150
29	ω_{232}	0.0066	(−1, −2)	−3	0	0	0.6150	
30	ω_{27}	0.0061	0	(+1,+2)	0	0		0.2168
31	ω_{32}	0.0061	0	0	(−1, −2)	0	0.2168	
32	ω_5	0.0058	0	0	0	+1		0.5993
33	ω_6	0.0058	−1	0	0	0	0.5993	
Least frequently visited states in equilibrium (bottom 6)								
210	ω_{284}	0.000007	+1	0	−2	−3	0.3035	0.0000
211	ω_{355}	0.000007	+3	+2	0	−1	0.0000	0.3035
212	ω_{146}	0.000005	0	+2	0	(−1, −3)	0.0000	0.3035
213	ω_{245}	0.000005	(+1,+3)	0	−2	0	0.3035	0.0000
214	ω_{300}	0.000001	−1	0	−2	−3	0.5993	
215	ω_{351}	0.000001	+3	+2	0	+1		0.5993
							Conditional averages	
							0.3775	0.3775

Average bid–ask spread	**Number of hit states (both sides of the book nonempty)**
1.0699	70

Probability that the book is nonempty on both sides
9.55%

Buy				Sell				Total
P_1	P_2	P_3	P_4	P_1	P_2	P_3	P_4	
Average conditional execution probabilities of limit orders								
0.1233	0.3994	0.6337	0.8780	0.8780	0.6337	0.3994	0.1233	
Percentages of submitted orders								
5.82%	20.62%	17.47%	6.10%	6.10%	17.47%	20.62%	5.82%	100.00%
Limit buy orders:			30.53%	**Limit sell orders:**			30.53%	
Market buy orders:			19.47%	**Market sell orders:**			19.47%	

$u \in (\theta_L^b(s_t), B]$, in any state s_t, where the ask side is nonempty, i.e., $p_L = a(s_t)$. Symmetrically, traders with extremely low private values submit market sell orders, $d_K^{*s}(s_t) = 1$ if and only if $u \in [A, \theta_K^s(s_t)]$, in any state s_t, where the bid side is nonempty, i.e., $p_K = b(s_t)$.

To determine the percentage of market order submissions on each side of the book (buy or sell) in the equilibrium of our model, we calculate the probability of a market order submission (for buy and sell sides separately) in each state, and then calculate a weighted average of these probabilities across all states (where the weight for each state is the equilibrium probability of that state) for buy and sell sides respectively (see under the section "Percentages of submitted orders" in Tables 1–4). In other words, we calculate the unconditional probability of observing a market order on each side of the book. In the limit order market equilibria shown in Tables 1–4, we find that as the dispersion in traders' private valuations increases (as w increases for any given level of M), the equilibrium percentage of market order submissions increases (from 34.90% in Table 1 to 35.70% in Table 2 and from 37.34% in Table 3 to 38.94% in Table 4) as well, since traders with extreme private values tend to follow more aggressive order placement strategies and demand more liquidity in equilibrium.[36]

Further, in our model specifications in Tables 1–4, given the probability of a market buy and/or a market sell order submission in each state, we compute equally weighted conditional averages of market buy and market sell order submission probabilities (averaged across all states where the ask (bid) side is nonempty), and report them for the buy and sell sides respectively. For each level of M, one can note that the conditional averages of market buy and market sell order probabilities also increase with the dispersion (w) in traders' private valuations. For example, in the equilibrium with $M = 2$ and $w = 1$ ($M = 3$ and $w = 1$), conditional on the ask side of the limit order book being nonempty, the average probability of observing a market buy order submission is 0.3851 (0.3726). In the equilibrium with $M = 2$ and $w = 4$ ($M = 3$ and $w = 4$), conditional on the ask side of the limit order book being nonempty, the average probability of observing a market buy order submission is 0.4026 (0.3775).

6. Empirical implications

Our model has a number of new empirical implications about the dynamic evolution of the limit order book, the bid–ask spreads, the effect of the price and time priority rule on the optimal placement of limit orders, and the decomposition of order flow between market orders and limit orders.

The main empirical prediction of our paper follows from our comparative dynamics result that bid–ask spreads in a limit order market are increasing in the dispersion in private values across traders. When the dispersion of agents' private values is small, our model predicts that submitting limit buy or sell orders at price quotes far from the middle point of the limit order book is less profitable. Since, in this case, agents (on both sides of the book) are more patient, their demand for liquidity is lower, i.e., their tendency to submit more aggressive limit orders and market orders is not strong. Dynamically, this implies that the future execution probabilities of more conservative limit orders submitted (on the other side of the book) in the current time period are lower. Hence, the expected returns to placing buy (sell) limit orders that are far below (above) an investor's own private valuation are lower even though potential gains from limit orders conditional on execution are high. On the other hand, as the dispersion in agents' private values increases, the number of impatient traders with a higher liquidity demand increases, which makes it more profitable for other traders to place limit orders that are more conservative with larger potential profits since the execution probability of these orders increases with the presence of more aggressive traders demanding liquidity. Thus, under market conditions with a large dispersion in liquidity traders' private values, the expected returns to placing buy (sell) limit orders that are far below (above) an investor's own private valuation are also higher in equilibrium. Hence, a wider distribution of private values leads to more order placement at prices away from the consensus value, and therefore, to a larger bid–ask spread. However, since private values of traders are not directly observable, empirical proxies for the dispersion in traders' private values are needed to test this prediction. Two proxies for the dispersion in private values may be the share turnover and

[36] Note that since the empty limit order book, state ω_1, is less frequently observed in equilibrium when $M = 3$ than when $M = 2$, the unconditional probability of observing market order submissions is higher when $M = 3$.

the abnormal trading volume (see Harris and Raviv, 1993) or the conditional volatilities of these two measures.[37]

According to the equilibrium results that we obtain from numerical solutions to our model, we also predict that the equilibrium order flow depends on the current state of the limit order book in the sense that an agent's optimal trading strategy is largely affected by the time and price priorities of the existing limit orders in the book. Therefore, order undercutting is a prediction that comes out endogenously from our model. Next, our model predicts that for a given level of dispersion in agents' private valuations, the average bid–ask spread will be smaller if limit orders have longer life spans. Finally, our model predicts that the percentage of market orders in the equilibrium order flow is also increasing with the dispersion in agents' private valuations.

7. Conclusion

In this paper, we developed a dynamic model of a limit order market populated with liquidity traders who have only private values. We characterized and analyzed the equilibrium order placement strategies of traders and the conditional execution probabilities of limit orders as a function of traders' liquidity demand and the state of the limit order book. We solved for the equilibrium of the model numerically, and analyzed its properties by performing comparative dynamics analysis. Our analysis showed that changes in the steady state of the limit order book and optimal order placement strategies reflect corresponding changes in the trade-off between order execution risk and the size of potential trading gains. We demonstrated how changes in the dispersion of traders' private values affect agents' optimal trading strategies and conditional execution probabilities of limit orders. Our main result is that the dispersion in private values across traders has a significant impact on the stationary state of the equilibrium limit order book and the average bid–ask spread. In our numerical experiments, we showed that when the dispersion of agents' private valuations of the asset is small, submitting limit buy or sell orders at price quotes far from the middle point of the limit order book is less profitable. On the other hand, as the dispersion in agents' private values for the asset increases, the number of impatient traders with a higher liquidity demand increases. This in turn makes it more profitable for liquidity traders with moderate private values to place more conservative limit orders with larger potential profits, since these orders are more likely to be executed due to the increased presence of more aggressive traders demanding liquidity. Thus, a wider distribution of private values leads to more order placement at prices away from the consensus value, and therefore, to a larger bid–ask spread. Further, our numerical simulations showed that extending the life span of limit orders reduces the average bid–ask spread observed in equilibrium. Finally, we found that the equilibrium percentage of market order submissions is also increasing in the dispersion in liquidity traders' private values.

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[37] In Harris and Raviv (1993), difference in opinions about the value of an asset leads to trade. While, in our model, the difference in valuations are driven purely by differences in private valuations (there is no heterogeneity in beliefs in our setting), a greater dispersion in private valuations may also lead to trade (with investors having low private valuations selling the asset to those with high private valuations). Note that a greater dispersion in private valuations may lead liquidity traders with more extreme values to follow more aggressive order placement strategies. Thus, there will be a greater volume of trade when the dispersion in private valuations across investors is greater.

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