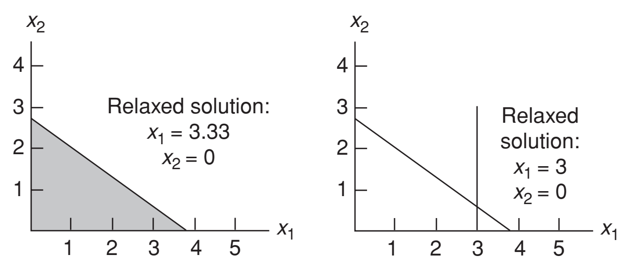
Chapter Five: Integer Programming

PROBLEM SUMMARY

1. Integer model, graphical solution
2. Integer model
3. Integer model
4. Integer model
5. Integer model
6. Integer model
7. Integer model
8. Mixed integer model
9. 0–1 integer model
10. Integer model
11. Integer model
12. 0–1 integer model
13. Integer model, formulation and computer solution
14. Integer model, formulation and computer solution
15. Set covering problem, integer model, formulation and computer solution
16. Integer model, formulation and computer solution
17. Integer model, formulation and computer solution
18. Integer model, knapsack problem, formulation and computer solution
19. Integer model, formulation and computer solution
20. Plant location problem, integer model, formulation and computer solution
21. Continuation of Problem 20, formulation
22. Continuation of Problem 20, formulation
23. Integer model, formulation and computer solution
24. 0–1 integer-model, computer solution
25. Integer model, computer solution
26. Integer model, formulation and computer solution
27. Integer model with 0-1 restriction
28. Integer model, formulation and computer solution
29. 0–1 integer model, computer solution
30. Integer model, formulation and computer solution
31. Mixed-integer model, formulation and computer solution (7–30)
32. Set covering problem, 0–1 integer model and computer solution
33. 0–1 integer model and computer solution
34. Set covering problem, 0–1 integer model and computer solution
35. Facility location problem, 0–1 integer model and computer solution

PROBLEM SOLUTIONS

1.



2. $x_1 = 6, x_2 = 0, Z = 18$

3. **(a)** maximize $Z = 50x_1 + 40x_2$ (profit)

subject to

$$3x_1 + 5x_2 \leq 150 \text{ yd}^2$$
$$10x_1 = 4x_2 \leq 200 \text{ hr.}$$
$$x_1, x_2 \geq 0 \text{ and integer}$$

(b) Relaxed solution:

$$x_1 = 10.5, x_2 = 23.7, Z = 1{,}473$$

Rounded down solution:

$$x_1 = 10, x_2 = 23, Z = 1{,}420$$

Integer solution:

$$x_1 = 10, x_2 = 24, Z = 1{,}460$$

The rounded down solution is not optimal.

4. **(a)** maximize $Z = \$400x_1 + 100x_2$

subject to

$$\begin{aligned} 8x_1 = 10x_2, &\le 80 \\ 2x_1 = 6x_2 &\le 36 \\ x_1 &\le 6 \\ x_1, x_2 &\ge 0 \text{ and integer} \end{aligned}$$

(b) Relaxed solution:

$$x_1 = 6, x_2 = 3.2, Z = 2{,}720$$

Rounded down solution:

$$x_1 = 6, x_2 = 3, Z = 2{,}700$$

Integer solution:

$$x_1 = 6, x_2 = 3, Z = 2{,}700$$

Integer solution same as rounded down solution.

5. **(a)** maximize $Z = 50x_1 + 10x_2$

subject to

$$\begin{aligned} x_1 + x_2, &\le 15 \\ 4x_1 + x_2 &\le 25 \\ x_1, x_2 &\ge 0 \text{ and integer} \end{aligned}$$

(b) $x_1 = 6, x_2 = 1, Z = 310$

6. **(a)** maximize $Z = 600x_1 + 540x_2 + 375x_3$

subject to

$$\begin{aligned} x_1 + x_2 + x_3, &\le 12 \\ x_1 &\le 5 \\ 80x_1 + 70x_2 + 50x_3 &\le 750 \\ x_1, x_2, x_3 &\ge 0 \text{ and integer} \end{aligned}$$

(b) $x_1 = 0, x_2 = 10, x_3 = 1, Z = 5{,}775$

7. **(a)** maximize $Z = 50x_1 + 40x_2$

subject to

$$\begin{aligned} 2x_1 + 5x_2, &\le 35 \\ 3x_1 + 2x_2 &\le 20 \\ x_1, x_2 &\ge 0 \text{ and integer} \end{aligned}$$

(b) Relaxed solution:

$$x_1 = 2.73, x_2 = 5.91, Z = 372.9$$

Rounded down solution:

$$x_1 = 2, x_2 = 5, Z = 300$$

Integer solution:

$$x_1 = 4, x_2 = 4, Z = 360$$

The rounded down solution is not optimal.

8. **(a)** maximize $Z = \$8000x_1 + 6000x_2$

subject to

$$\begin{aligned} 70x_1 + 30x_2, &\le 500 \\ x_1 + 2x_2 &\le 14 \\ x_1 &\ge 0 \text{ and integer} \\ x_2 &\ge 0 \end{aligned}$$

(b) $x_1 = 5, x_2 = 4.5, Z = 67{,}000$

9. $x_1 = 1, x_2 = 0, x_3 = 1, Z = 1{,}800$

10. $x_1 = 0, x_2 = 4, x_3 = 1.33, Z = 29.32$

11. minimize $Z = 81x_1 + 50x_2$

subject to

$$\begin{aligned} 76x_1 = 53x_2, &\ge 600 \\ x_1 + x_2 &\le 10 \\ 1.3x_1 = 4.1x_2, &\le 24 \\ x_1, x_2 &\ge 0 \text{ and integer} \end{aligned}$$

Solution:

$$x_1 = 6$$
$$x_2 = 3$$
$$Z = \$636$$

12. $x_1 = 1, x_4 = 1, Z = 60$

13. a. Maximize $Z = 85{,}000x_1 + 60{,}000x_2 - 18{,}000y_1$

subject to

$$\begin{aligned} x_1 + x_2 &\le 10 \\ 10{,}000x_1 + 7{,}000x_2 &\le 72{,}000 \\ x_1 - 10y_1 &\le 0 \\ x_1, x_2 &\ge 0 \text{ and integer} \\ y_1 &= 0 \text{ or } 1 \end{aligned}$$

b. $x_1 = 0, x_2 = 10, y_1 = 0, Z = \$600{,}000$

14. a. Maximize $Z = \$.36x_1 + .82x_2 + .29x_3 + .16x_4 + .56x_5 + .61x_6 + .48x_7 + .41x_8$

subject to

$$60x_1 + 110x_2 + 53x_3 + 47x_4 + 92x_5 + 85x_6 + 73x_7 + 65x_8 \leq 300$$
$$7x_1 + 9x_2 + 8x_3 + 4x_4 + 7x_5 + 6x_6 + 8x_7 + 5x_8 \leq 40$$
$$x_2 - x_5 \leq 0$$
$$x_i = 0 \text{ or } 1$$

b. $Z = \$1.99$ million; $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$, $x_5 = 1$, $x_6 = 1$, $x_7 = 0$

15. a. x_i = no. of employees assigned to time period i, $i = 1, 2, \ldots, 6$ (time period 1 = 12:00 midnight–4:00 A.M.; period 2 = 4:00–8:00 A.M.; etc.)

minimize $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

subject to

$$x_6 + x_1 \geq 90$$
$$x_1 + x_2 \geq 215$$
$$x_2 + x_3 \geq 250$$
$$x_3 + x_4 \geq 65$$
$$x_4 + x_5 \geq 300$$
$$x_5 + x_6 \geq 125$$
$$x_i \geq 0$$

b. $x_1 = 90$, $x_2 = 250$, $x_3 = 0$, $x_4 = 175$, $x_5 = 125$, $x_6 = 0$, $Z = 640$

16. x_1 = day contacts by phone
x_2 = day contacts in person
x_3 = night contacts by phone
x_4 = night contacts in person

Maximize $Z = \$16x_1 + 33x_2 + 17x_3 + 37x_4$

subject to:

$$x_2 + x_4 \leq 575$$
$$6x_1 + 13x_2 \leq 1{,}320$$
$$7x_3 + 19x_4 \leq 2{,}580$$
$$x_1, x_2, x_3, x_4 \geq 0 \text{ and integer}$$

Integer solution:

$x_1 = 220$
$x_3 = 368$
$Z = \$9{,}776$

The non-integer solution is:

$x_1 = 220$
$x_3 = 368.57$
$Z = \$9{,}785.71$

The rounded down solution is only slightly less (i.e., \$9.71)

17. (a) x_1 = tv ads
x_2 = newspaper ads
x_3 = radio ads

minimize $Z = \$25{,}000x_1 + 7{,}000x_2 + 9{,}000x_3$

subject to:

$$53{,}000x_1 + 30{,}000x_2 + 41{,}000x_3 \geq 200{,}000$$
$$\frac{32{,}000x_1 + 20{,}000x_2 + 18{,}000x_3}{(21{,}000x_1 + 10{,}000x_2 + 23{,}000x_3)} \geq 1.5$$
$$\frac{34{,}000x_1 + 12{,}000x_2 + 24{,}000x_3}{(53{,}000x_1 + 30{,}000x_2 + 41{,}000x_3)} \geq .60$$
$$x_1, x_2, x_3, x_4 \geq 0 \text{ and integer}$$

Integer solution:

$x_1 = 4$
$x_2 = 0$
$x_3 = 0$
$Z = \$99{,}999.99$

(b) Non-integer solution:

$x_1 = 2.9275$
$x_2 = .9713$
$x_3 = .383$
$Z = \$83{,}433.65$

18. Maximize $Z = 90x_1 + 150x_2 + 30x_3$

subject to

$$2x_1 + 3x_2 + x_3 \leq 5$$

Solution: $Z = \$240$, $x_1 = 1$, $x_2 = 1$, $x_3 = 0$

19. x_1 = no. of salespeople to East, x_2 = no. of salespeople to Midwest, x_3 = no. of salespeople to West

maximize $Z = 25{,}000x_1 + 18{,}000x_2 + 31{,}000x_3$

subject to

$$x_1 + x_2 + x_3 = 100$$
$$5{,}000x_1 + 11{,}000x_2 + 7{,}000x_3 \leq 700{,}000$$
$$x_1 \geq 10$$
$$x_2 \geq 10$$
$$x_3 \geq 10$$
$$x_1, x_2, x_3 \geq 0 \text{ and integer}$$

Solution: $x_1 = 20$, $x_2 = 10$, $x_3 = 70$, $Z = 2{,}850{,}000$

20. x_{ij} = vehicles [1,000s shipped from plant i (i = 1, 2, 3, 4, 5) to warehouse j (j = A, B, C, D), y_i = plant i (i = 1, 2, 3, 4, 5) = 0 or 1

minimize $Z = 2{,}100y_1 + 850y_2 + 1{,}800y_3 + 1{,}100y_4 + 900y_5 + 56x_{1A} + 21x_{1B} + 32x_{1C} + 65x_{1D} + 18x_{2A} + 46x_{2B} + 7x_{2C} + 35x_{2D} + 12x_{3A} + 71x_{3B} + 41x_{3C} + 52x_{3D} + 30x_{4A} + 24x_{4B} + 28x_{4D} + 45x_{5A} + 50x_{5B} + 26x_{5C} + 31x_{5D} + 61x_{4C}$

subject to

$$c_1 - x_{1A} - x_{1B} - x_{1C} - x_{1D} = 0$$
$$c_2 - x_{2A} - x_{2B} - x_{2C} - x_{2D} = 0$$
$$c_3 - x_{3A} - x_{3B} - x_{3C} - x_{3D} = 0$$
$$c_4 - x_{4A} - x_{4B} - x_{4C} - x_{4D} = 0$$
$$c_5 - x_{5A} - x_{5B} - x_{5C} - x_{5D} = 0$$
$$x_{1A} + x_{2A} + x_{3A} + x_{4A} + x_{5A} = 6{,}000$$
$$x_{1B} + x_{2B} + x_{3B} + x_{4B} + x_{5B} = 14{,}000$$
$$x_{1C} + x_{2C} + x_{3C} + x_{4C} + x_{5C} = 8{,}000$$
$$x_{1D} + x_{2D} + x_{3D} + x_{4D} + x_{5D} = 10{,}000$$
$$c_1 \le 12{,}000y_1$$
$$c_2 \le 18{,}000y_2$$
$$c_3 \le 14{,}000y_3$$
$$c_4 \le 10{,}000y_4$$
$$c_5 \le 16{,}000y_5$$

Solution: $y_2, y_4, y_5 = 1$, $x_{2A} = 6{,}000$
$x_{2B} = 4{,}000$, $x_{2C} = 2{,}000$, $x_{4B} = 10{,}000$,
$x_{5C} = 6{,}000$, $x_{5D} = 10{,}000$, $Z = \$3{,}902{,}000$

21. Add the constraint $y_2 + y_4 \le 1$.

Solution: $y_2, y_3, y_5 = 1$, $x_{2B} = 14{,}000$
$x_{2C} = 2{,}000$, $x_{3A} = 6{,}000$, $x_{5C} = 6{,}000$,
$x_{5D} = 10{,}000$, $Z = \$4{,}786{,}000$

22. Add the constraint $y_5 \le y_1$.

Solution: $y_1, y_2, y_5 = 1$, $x_{1B} = 12{,}000$
$x_{2A} = 6{,}000$, $x_{2B} = 2{,}000$, $x_{2C} = 2{,}000$,
$x_{5C} = 6{,}000$, $x_{5D} = 10{,}000$, $Z = \$4{,}822{,}000$

23. Maximize $Z = 12{,}100x_1 + 8{,}700x_2 + 10{,}500x_3$

subject to:

$$360x_1 + 375x_2 + 410x_3 \le 30{,}000$$
$$x_1 + x_2 + x_3 \le 67$$
$$14x_1 + 10x_2 + 18x_3 \le 2{,}200$$
$$x_1/x_3 \ge 2$$
$$x_2/x_1 \ge 1.5$$
$x_1, x_2, x_3 \ge 0$ and interger

Integer solution:
$x_1 = 22$
$x_2 = 34$
$x_3 = 11$
$Z = \$677{,}500$

24. **a)** minimize $Z = 5x_1 + 10x_2 + 8x_3 + 12x_4 + 7x_5 + 10x_6 + 8x_7$

subject to

$$\frac{9x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 + 3x_6 + 9x_7}{3(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)} \ge 2.00$$
$$3(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7) \ge 12$$
$$x_2 + x_3 + x_4 + x_6 \le 2$$
$$x_1 + x_2 + x_6 + x_7 \ge 3$$
$$x_i = 0 \text{ or } 1$$

b) $x_1 = 1$ (Management I)
$x_2 = 1$ (Principles of Accounting)
$x_5 = 1$ (Marketing Management)
$x_7 = 1$ (English Literature)
$Z = 30$ hours per week
Minimum grade point average = 2.50

25. **a)** maximize $Z = 1{,}650x_1 + 850x_2 + 790x_3$

subject to

$6.3x_1 + 3.9x_2 + 3.1x_3 \le 125$
$17x_1 + 10x_2 + 7x_3 \le 320$
$x_1, x_2, x_3, \ge 0$ and integer

b) $x_1 = 10$
$x_3 = 20$
$Z = 32{,}300$
The relaxed, noninteger solution is,

$x_1 = 13.61$
$x_3 = 12.67$
$Z = 32{,}460.46$
The rounded down solution is $x_1 = 13$, $x_3 = 12$, and $Z = 30{,}930$, which is not optimal.

26. maximize $Z = 575x_1 + 120x_2$

subject to

$40x_1 + 15x_2 \le 600$
$30x_1 + 18x_2 \le 480$
$4x_1 - x_2 \le 0$
$x_1, x_2 \ge 0$ and integer

Optimal solution:

$x_1 = 4$
$x_2 = 20$
$Z = 4{,}700$

27. Maximize $Z = \$575x_1 + 120x_2 + 45x_3$

subject to:

$$40x_1 + 15x_2 + 4x_3 \leq 600$$
$$30x_1 + 18x_2 + 5x_3 \leq 480$$
$$4x_1 - x_2 \leq 0$$
$$x_3 = 20y_1$$
$$x_1, x_2, x_3 \geq 0 \text{ and interger}$$
$$y_1 = 0 \text{ or } 1$$

Or the last restriction that $y_1 = 0$ or 0 can be included in the model as a constraint, $y_1 \leq 1$.

Solution:

$x_1 = 3$
$x_2 = 16$
$x_3 = 20$
$y_1 = 1$
$Z = \$4,745$

They should produce the batch of 20 stools since the profit is slightly greater (\$4,745 vs. \$4,700).

28. x_1 = bass boat
x_2 = ski boat
x_3 = speed boat

Maximize $Z = 20,500x_1 + 12,000x_2 + 22,300x_3$

subject to:

$$1.3x_1 + 1.0x_2 + 1.5x_3 \leq 210$$

$$\frac{x_1}{(x_2 + x_3)} \leq 2$$

$$x_1 + 2x_3 \leq 160$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer}$$

Solution:

$x_1 = 110$
$x_2 = 31$
$x_3 = 24$
$Z = \$3,162,200$

29. a. maximize $Z = 18x_{1A} + 20x_{1B} + 21x_{1C} + 17x_{1D} + 19x_{2A} + 15x_{2B} + 22x_{2C} + 18x_{2D} + 20x_{3A} + 20x_{3B} + 17x_{3C} + 19x_{3D} + 24x_{4A} + 21x_{4B} + 16x_{4C} + 23x_{4D} + 22x_{5A} + 19x_{5B} + 21x_{5C} + 21x_{5D}$

subject to

$(.3x_{1A} + .9x_{1B} + .6x_{1C} + .4x_{1D} + .8x_{2A} + .5x_{2B} + 1.1x_{2C} + .7x_{2D} + 1.1x_{3A} + 1.3x_{3B} + .6x_{3C} + .8x_{3D} + 1.2x_{4A} + .8x_{4B} + .6x_{4C} + .9x_{4D} + 1.0x_{5A} + .9x_{5B} + 1.0x_{5C} + 1.0x_{5D})/(18x_{1A} + 20x_{1B} + 21x_{1C} + 17x_{1D} + 19x_{2A} + 15_{2B} + 22x_{2C} + 18x_{2D} + 20x_{3A} + 20x_{3B} + 17x_{3C} + 19x_{3D} + 24x_{4A} + 21x_{4B} + 16x_{4C} + 23x_{4D} + 22x_{5A} + 1\ 9x_{5B} + 21x_{5C} + 21x_{5D}) \leq .04$

$$x_{1A} + x_{1B} + x_{1C} + x_{1D} \leq 1$$
$$x_{2A} + x_{2B} + x_{2C} + x_{2D} \leq 1$$
$$x_{3A} + x_{3B} + x_{3C} + x_{3D} \leq 1$$
$$x_{4A} + x_{4B} + x_{4C} + x_{4D} \leq 1$$
$$x_{5A} + x_{5B} + x_{5C} + x_{5D} \leq 1$$

$$x_{1A} + x_{2A} + x_{3A} + x_{4A} + x_{5A} = 1$$
$$x_{1B} + x_{2B} + x_{3B} + x_{4B} + x_{5B} = 1$$
$$x_{1C} + x_{2C} + x_{3C} + x_{4C} + x_{5C} = 1$$
$$x_{1D} + x_{2D} + x_{3D} + x_{4D} + x_{5D} = 1$$
$$x_{ij} = 0 \text{ or } 1$$

b) $x_{1C} = 1$
$x_{3D} = 1$
$x_{4B} = 1$
$x_{5A} = 1$
Z = 83 parts

30. Minimize $Z = 120x_1 + 75x_2$

subject to:

$$220x_1 + 140x_2 \leq 6,300$$
$$x_1 + x_2 \leq 32$$
$$.4x_1 + .9x_2 \leq 15$$
$$x_1, x_2 \geq 0 \text{ and interger}$$

Non-integer solution:

$x_1 = 25.1409$
$x_2 = 5.493$
$Z = \$3,428.87$

Integer solution:

$x_1 = 28$
$x_2 = 1$
$Z = \$3,435$